DEA Duality on Returns to Scale(RTS) in Production and Cost Analyses : An Occurrence of Multiple Solutions and Differences between Production-based and Cost-based RTS Estimates

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1. Introduction

Data Envelopment Analysis(DEA) has been widely applied to the efficiency measurement of many decisional entities in public and private sectors. In the previous DEA studies, this research finds many DEA research efforts on Returns To Scale(RTS). Comparing these previous studies, this article finds a research need to explore a theoretical linkage, or so-called " duality," between production-based and cost-based DEA/RTS measurements, focusing uppon when multiple solutions occur on RTS and how to deal with such a difficulty.

2. DEA formulation

To describe the analytical structure of DEA/RTS measurement, this study first assumes that there are n DMUs (Decision Making Units), which is denoted by $j \in J = \{j \mid j = 1, 2, \ldots, n\}$. The production process of each DMU is characterized by both an input column vector $X_j = (x_{1j}, x_{2j}, \ldots, x_{mj})^T > 0$ and an output column vector $Y_j = (y_{1j}, y_{2j}, \ldots, y_{sj})^T > 0$. Under these assumptions, DEA production analysis is formulated by following models:

$$
\begin{aligned}
\min \quad & \theta \\
\text{s.t.} \quad & -\sum_{j=1}^{n} x_{ij}\lambda_j + \theta x_{ik} \geq 0, \quad i = 1, \ldots, m \\
& \sum_{j=1}^{n} y_{rj}\lambda_j \geq y_{rk}, \quad r = 1, \ldots, s \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0 \text{ and } \theta \geq 0.
\end{aligned} \tag{1}
$$

Here, variables $\lambda_j (j = 1, \ldots, n)$ are used to make a convex hull of inputs and outputs in these data spaces. The optimal θ^* scores of Model (1) indicates the level of a production-based DEA efficiency score. The efficiency score measured by Model(1) is conventionally referred to as " Technical Efficiency(TE)" in production economics.

Besides the input and output vectors used for production analysis, this study assumes that we can access an input price vector $P_j = (p_{1j}, p_{2j}, \ldots, p_{mj})$. Under such an additional assumption, we can also formulate DEA cost analysis. After obtaining the cost minimum, the level of cost-based efficiency(η_k^*) of the k^{th} DMU is determined by

$$
\eta_k^* = \sum_{i=1}^{m} p_{ik} x_i^* \Big/ c_k = \sum_{i=1}^{m} p_{ik} x_i^* \Big/ \sum_{i=1}^{m} p_{ik} x_{ik}.
$$

The efficiency is determined by the ratio of the cost minimum to the observed cost. This type of efficiency is referred to as " Overall Efficiency (OE)" .

To extend (1) further into the RTS measurement, this study needs to document its dual formulation that is mathematically expressed by Model (2).

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} w_r y_{rk} + \sigma_1 - \sigma_2 \\
\text{s.t.} \quad & -\sum_{i=1}^{m} v_i x_{ij} + \sum_{r=1}^{s} w_r y_{rj} + \sigma_1 - \sigma_2 \leq 0, \quad j = 1, \ldots, n \\
& \sum_{i=1}^{m} v_i x_{ik} = 1, \\
& v_i \geq 0, w_r \geq 0, \sigma_1 \geq 0, \text{and } \sigma_2 \geq 0.
\end{aligned} \tag{2}
$$

The dual variable $\sigma (= \sigma_1 - \sigma_2)$ corresponds to the intercept of a supporting hyperplane on the k^{th} DMU.

3. Optimal Conditions

Optimal conditions between (1) and (2) can be summarized in the following manner:

$$
\begin{aligned}
&(a)\ \theta^* = \sum_{r=1}^{s} w_r^* y_{rk} + \sigma_1^* - \sigma_2^*, \\
&(b)\ d_i^{x*} v_i^* = 0,\ d_i^{x*} + v_i^* > 0, i = 1, \ldots, m, \\
&(c)\ d_r^{y*} w_r^* = 0,\ d_r^{y*} + w_r^* > 0, r = 1, \ldots, s, \text{and} \\
&(d)\ \lambda_j^* t_j^* = 0,\ \lambda_j^* + t_j^* > 0,\ j = 1, \ldots, n.
\end{aligned} \tag{3}
$$

where

$$
d_i^x = \theta x_{ik} - \sum_{j=1}^{n} x_{ij}\lambda_j (i = 1, \ldots, m),\ d_r^y = \sum_{j=1}^{n} y_{rj}\lambda_j - y_{rk}
$$

$(r = 1, ..., s)$ and $t_j = -\sum_{i=1}^{m} v_i x_{ij} + \sum_{r=1}^{s} w_r y_{rj} + \sigma_1 - \sigma_2$.

4. Characterization of a Supporting Hyperplane

Three types of RTS are determined by examining a sign of the intercept (σ) of a supporting hyperplanes. Hence, before discussing RTS, this study needs to characterize a mathmatical structure of the supporting hyperplanes.

Proposition 1: A production-based supporting hyperplane on the k[th] DMU is mathematically expressed by

$$-\sum_{i=1}^{m} v_i x_i + \sum_{r=1}^{s} w_r y_r + \sigma_1 - \sigma_2 = 0 \qquad (4)$$

where, the parameter $\sigma (= \sigma_1 - \sigma_2)$ indicates the intercept of the supporting hyperplane.

Proposition 2: A cost-based supporting hyperplane on the k[th] DMU is mathematically expressed by

$$c = \sum_{r=1}^{s} w_r y_r + \sigma_1 - \sigma_2 \qquad (5)$$

where, the parameter $\sigma (= \sigma_1 - \sigma_2)$ indicates the intercept of the supporting hyperplane.

5. An Occurrence of Multiple Solutions

Multiple solutions may occur on the intercept (σ) of a supporting hyperplanes. The occurrence of such multiple solutions (multiple RTS) in production-based DEA/RTS measurement can be mathematically characterized by the following condition:

Proposition 3: After we measure the efficiency of the k[th] DMU, using Model (1), we compute the following numbers;

$\#(R_k)$ = the number of $\{j \mid \lambda_j^* > 0,\ j = 1, ..., n\}$,

$\#(D_k^x)$ = the number of $\{i \mid d_i^{x*} > 0,\ i = 1, ..., m\}$ and

$\#(D_k^y)$ = the number of $\{r \mid d_r^{y*} > 0,\ r = 1, ..., s\}$.

Let $\alpha_k = m + s - \#(D_k^x) - \#(D_k^y) - \#(R_k)$,

(a) if $\alpha_k > 0$, then (4) has an infinite number of optimal solutions,

(b) if $\alpha_k = 0$, then (4) has a unique optimal solution,

(c) α_k is never negative.

The occurrence of multiple RTS in cost-based DEA/RTS measurement can be characterized by the following condition:

Proposition 4: After we measure the OE_V of the k[th] DMU, we compute $\#(R_k)$ and $\#(D_k^y)$.

Let $\beta_k = s + 1 - \#(D_k^y) - \#(R_k)$,

(a) if $\beta_k > 0$, then (5) has an infinite number of optimal solutions,

(b) if $\beta_k = 0$, then (5) has a unique optimal solution,

(c) β_k is never negative.

6. How to Deal with Multiple Solution

Returning to Model(2), this research estimates the upper and lower bounds of σ by the following formulation:

$$\begin{aligned} &\text{max/min} \quad \sigma_1 - \sigma_2 \\ &\text{s.t.} \quad -\sum_{i=1}^{m} v_i x_{ij} + \sum_{r=1}^{s} w_r y_{rj} + \sigma_1 - \sigma_2 \le 0, \quad j \in J - IE \\ &\qquad \sum_{i=1}^{m} v_i x_{ik} = 1, \qquad (6) \\ &\qquad \sum_{r=1}^{s} w_r y_{rk} + \sigma_1 - \sigma_2 = \theta_V^*, \\ &\qquad v_i \ge 0, w_r \ge 0, \sigma_1 \ge 0, \text{and}\ \sigma_2 \ge 0. \end{aligned}$$

An important feature of above model is that it incorparates the level of $TE_V(\theta_V^*)$ into (6). IE indicates a group of inefficient DMUs. For cost-based DEA/RTS measurement, we incorporate the cost minimum (c_k^*) into the right hand side of $\sum_{r=1}^{s} w_r y_{rj} + \sigma_1 - \sigma_2$ in (6).

7. RTS Determination

Let $\overline{\sigma}^* (= \sigma_1^* - \sigma_2^*)$ be the optimal objective of the maximization problem of (6) and $\underline{\sigma}^*$ be that of its minimization. Then, the production (input)-based RTS estimate can be classified into the following three cases:

(a) IRTS $\leftrightarrow \underline{\sigma}^* > 0$, (b) CRTS $\leftrightarrow \overline{\sigma}^* \ge 0 \ge \underline{\sigma}^*$, or

(c) DRTS $\leftrightarrow 0 > \overline{\sigma}^*$ (7)

The above three cases (7) describe a mathematical description on the three types of RTS.

8. Conclusion

This study analytically characterizes a supporting hyperplane on a reference set that is measured under variable RTS technology and then proposes an arithmetric method to find an occurrence of multiple RTS solutions. This article also documents how to deal with such a difficulty. Besides, this article describes the dual relation on scale efficiencies and that of scale economies providing a theoretical linkage between production-based and cost-based RTS measurements.