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State-Probability Vector Relationship between a Finite-Capacity Queue and an Infinite-Capacity Queue with MAPs

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1 Introduction

This article considers state-probability vector relationship between a finite-capacity queue and an infinite-capacity queue with MAPs (Markovian Arrival Processes), and presents an exact formula of the loss probability for MAP/GI/1/K queue in terms of the state-probability vector of the corresponding infinite-capacity queue, where K is the total capacity for the system.

Unfortunately, an explicit formula of the loss probability and the state-probability vector for MAP/GI/1/K queue is not known at present. In general it requires much computational effort to obtain the state-probability vector for MAP/GI/1/K queue. On the other hand, numerical algorithms to solve the infinite-capacity queues have been extensively studied. Hence, it may be useful to express the state-probability vector for the finite-queue and the loss probability using that for the corresponding infinite-capacity queue.

2 The MAP finite-capacity and infinite-capacity queues

The queue considered in this article is characterized as follows. Customers arrive to the queue according to an m-state MAP with representation (D_0, D_1). Here D_0 and D_1 are $m \times m$ matrices. The MAP is a versatile point process where our familiar arrival processes such as Poisson process, IPP(Interrupted Poisson Process), and MMPP(Markov Modulated Poisson Process) are included as special cases.

In the MAP, D, the sum of D_0 and D_1, is the infinitesimal generator of the underlying continuos-time Markov chain $\{J(t);\, t \geq 0\}$ which governs customer arrivals. Note that $De = \mathbf{o}$, where e denotes a column vector of ones. Let $\boldsymbol{\pi}$ be the steady state-probability vector of D such that

$$\boldsymbol{\pi} D = 0, \qquad \boldsymbol{\pi} e = 1. \tag{1}$$

Let λ denote the traffic intensity of the arrival process, then λ is given by

$$\lambda = \boldsymbol{\pi} D_1 e\ . \tag{2}$$

Let $N(t)$ denote the queue length at time t, including a customer in service and P_{Loss} be the probability that an arriving customer is lost. Customers accepted by the system are served by the single server on the FIFO (first-in-first-out) basis. The service time is generally distributed with probability distribution function of $B(x)$ with mean $E(B)$ and the Laplace-Stieltjes transform (LST) $\tilde{B}(s) = \int_0^\infty e^{-sx} B(dx)$. Let $\bar{B}(t)$ be the remaining service time of $B(t)$.

In this section, we perform an analysis to the queueing model by using the supplementary variable method. It is clear that the joint distribution of the queue length $N(t)$, state $J(t)$, and a supplementary variable $\bar{B}(t)$ has a Markovian property with $0 \leq N(t) \leq K$ and $1 \leq J(t) \leq m$.

We further define the following notations for our analysis.

$$\begin{aligned}
P_i(x) &= (P_{i,1}(x),\ \ldots,\ P_{i,m}(x)),\\
P_0 &= (P_{0,1},\ \ldots,\ P_{0,m}),\\
P_{i,j}(x)dx &= \lim_{t\to\infty} \Pr\{N(t) = i,\ J(t) = j,\\
&\qquad x < \bar{B}(t) < x + dx\},\\
P_{0,j} &= \lim_{t\to\infty} \Pr\{N(t) = 0,\ J(t) = j\}
\end{aligned}$$

where $i = 1, 2, \ldots, K$, $j = 1, \ldots, m$. We also define $P_i^{(\infty)}(x)$, $P_0^{(\infty)}$, $P_{i,j}^{(\infty)}(x)dx$, $P_{0,j}^{(\infty)}$, the joint distributions of the corresponding infinite-capacity queue in the same way. Observing the system state at time t and $t + \Delta$, we have the following equations.

$$0 = P_0 D_0 + P_1(0) \tag{3}$$

$$\frac{dP_1(x)}{dx} = -P_1(x)D_0 - P_0(x)D_1\frac{dB(x)}{dx} - P_2(0)\frac{dB(x)}{dx} \quad (4)$$

$$\frac{dP_i(x)}{dx} = -P_i(x)D_0 - P_{i-1}(x)D_1 - P_{i+1}(0)\frac{dB(x)}{dx} \quad (5)$$

$$\frac{dP_K(x)}{dx} = -P_K(x)D - P_{K-1}(x)D_1 \quad (6)$$

where $i = 2, \ldots, K-1$. We denote the LT vectors by

$$\tilde{P}_i(s) = (\tilde{P}_{i,1}(s), \ldots, \tilde{P}_{i,m}(s))$$
$$\tilde{P}_i^{(\infty)}(s) = (\tilde{P}_{i,1}^{(\infty)}(s), \ldots, \tilde{P}_{i,m}^{(\infty)}(s))$$

where $\tilde{P}_{i,j}(s)$ $(\tilde{P}_{i,j}^{(\infty)}(s))$is the LT of $P_{i,j}(x)$ $(P_{i,j}^{(\infty)}(x))$. We then obtain the Laplace transforms,

$$\tilde{P}_1(s)(sI + D_0) = \tilde{P}_1(0) - P_0D_1\tilde{B}(s) - \tilde{P}_2(0)\tilde{B}(s), \quad (7)$$

$$\tilde{P}_i(s)(sI + D_0) = \tilde{P}_i(0) - \tilde{P}_{i-1}(s)D_1 - \tilde{P}_{i+1}(0)\tilde{B}(s), \quad (8)$$

$$\tilde{P}_K(s)(sI + D) = \tilde{P}_K(0) - \tilde{P}_{K-1}(s)D_1, \quad (9)$$

where $i = 1, \ldots, K-1$. From the normalization condition,

$$P_0 e + \sum_{i=1}^{K} \tilde{P}_i(0)e = 1. \quad (10)$$

3 Result

THEOREM. The state-probability vector for MAP/GI/1/K queue is proportional to that for the corresponding infinite-capacity queue, and the following equations hold,

$$\tilde{P}_k(0) = \frac{1}{P_0^{(\infty)}e + \sum_{k=1}^{K-1} \tilde{P}_k^{(\infty)}(0)e + \tilde{P}_{K-1}^{(\infty)}(0)D_1 e}\tilde{P}_k^{(\infty)}(0), \quad (11)$$

where $k = 0, \ldots, K-1$. We can also express the loss probability P_{Loss} in terms of $\tilde{P}_k^{(\infty)}(0)$,

$$P_{Loss} = 1 - \frac{c}{\rho}\left(\sum_{k=1}^{K-1} \tilde{P}_k^{(\infty)}(0)e + \tilde{P}_{K-1}^{(\infty)}(0)D_1 e\right). \quad (12)$$

4 The proof of Theorem

In this section, we prove the above result. By applying Little's law to the service process of the server (excluding the waiting room), we have

$$P_0 e = 1 - (1 - P_{Loss})\rho, \quad (13)$$

where $\rho = \pi D_1 e E(B)$.

Note that (3)-(5) are identical with the corresponding equations for MAP/GI/1 queue with an infinite-capacity. Therefore we can show that the state-probability vector for MAP/GI/1/K queue is proportional to that for the corresponding infinite-capacity queue, and the following equations hold,

$$P_0 = cP_0^{(\infty)}, \; P_i(x) = cP_i^{(\infty)}(x), \quad (14)$$

where c is a constant and $i = 1, \ldots, K-1$. We then perform the Laplace transforms of (14), we obtain

$$\tilde{P}_i(s) = c\tilde{P}_i^{(\infty)}(s) \quad (1 \le i \le K-1). \quad (15)$$

Substituting $s = 0$ into (9) yields

$$\tilde{P}_K(0)e = \tilde{P}_{K-1}(0)D_1 e\,. \quad (16)$$

Substituting (14), (15), (16) into (10), we have

$$cP_0^{(\infty)}e + \sum_{k=1}^{K-1} c\tilde{P}_k^{(\infty)}(0)e + c\tilde{P}_{K-1}^{(\infty)}(0)D_1 e = 1, \quad (17)$$

from which we can determine the proportional constant c,

$$c = \frac{1}{P_0^{(\infty)}e + \sum_{k=1}^{K-1} \tilde{P}_k^{(\infty)}(0)e + \tilde{P}_{K-1}^{(\infty)}(0)D_1 e}. \quad (18)$$

From these results, we have shown that the equations (11) and (12) hold and we can express the state-probability vectors $\tilde{P}_k(0)$ and the loss probability P_{Loss} in terms of $\tilde{P}_k^{(\infty)}(0)$.

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