1998年度日本オペレーションズ・リサーチ学会
秋季研究発表会

An algorithm for solving the edge-disjoint path problem on tournament graphs

02401223 徳島大学 *中山慎一 NAKAYAMA Shin-ichi
01603863 豊橋技術科学大学 増山 繁 MASUYAMA Shigeru

1 Introduction

Given a connected graph $G = (V, E)$ and K pairs of vertices (x_i, y_i), $i = 1, \cdots, K$, the edge-disjoint path problem asks to construct K pairwise edge-disjoint paths connecting each pair (x_i, y_i) from *source* x_i to *sink* y_i, $i = 1, \cdots, K$, where paths $P_1, P_2, \cdots, P_l$, $l \geq 2$, are edge-disjoint. A *tournament graph* (*tournament* for short) is a directed graph such that there is precisely one edge between each pair of vertices. On tournaments, J. Bang-Jensen showed a necessary and sufficient condition for the existence of edge-disjoint (x_1, y_1)-, (x_2, y_2)-paths and an $O(n^4)$ time algorithm for examining the existence of such paths where n is the number of vertices [1]. In this paper, we propose an $O(n^2)$ time algorithm for examining the existence of edge-disjoint (x_1, y_1)-, (x_2, y_2)-paths and for constructing them, if they exist, using the property of tournaments.

2 Definition

A *digraph* D consists of a pair $V(D)$, $A(D)$ where $V(D)$ is a finite set of vertices and $A(D)$ is a set of ordered pairs (u, v) of vertices, called edges. If an edge (u, v) exists in $A(D)$, we say that u dominates v. The number of vertices $y \in U \subseteq V(D)$ dominated by x is denoted by $d^+_U(x)$. We call $d^+_{V(D)}(x)$ the *out-degree* of x and simply is denoted by $d^+(x)$. Similarly, the number of vertices $y \in U \subseteq V(D)$ dominating x is denoted by $d^-_U(x)$ and $d^-_{V(D)}(x)$ ($d^-(x)$ for short) is called the *in-degree* of x. A *component* D' of a digraph is a maximal subgraph such that for any two vertices x, y of D', D' contains an (x, y)-path and (y, x)-path. A digraph D is *strong* if it has only one component.

3 Algorithm

We first describe a property of tournament.

[Property 1] *When tournament T is not strong, it is divided into some components and we can label these components $T_1, T_2, \cdots, T_l$ such that each vertex of T_j dominates all vertices of T_i if $i < j$.* □

We say that T_1 (respectively, T_l) is the *initial component* (respectively, the *terminal component*) of T.

By Property 1, for each degree $d^+(v)$, if $v_{i'} \in V(T_i)$, $v_{j'} \in V(T_j)$, $i < j$, is satisfied, then $d^+(v_{i'}) < d^+(v_{j'})$ holds. Moreover, the following lemma is deduced.

[Lemma 1] *If $d^+(v_{i'}) = d^+(v_{j'})$ is satisfied, then $v_{i'}$, $v_{j'}$ belong to the same component.*

J. Bang-Jensen gave the necessary and sufficient condition of the existence of two edge-disjoint (x_1, y_1)-, (x_2, y_2)-paths in tournament T.

[Definition 1]*[1] Let T be a strong tournament and let x_1, y_1, x_2, y_2 be four different vertices in T. The 5-tuple (T, x_1, x_2, y_1, y_2) is said to be of*
Type 1a. There exists a proper subset $S_1 \subset V(T)$ such that $y_1, y_2 \in S_1$, $x_1, x_2 \in S_2 = T - S_1$ and there is exactly one edge from S_2 to S_1 in T.
Type 1b. It is not of Type 1a and there exists a partition S_1, S_2, S_3 of $V(T)$ into disjoint non empty subsets with the following conditions. $y_i \in S_1$, $x_i, y_{3-i} \in S_2$, $x_{3-i} \in S_3$, for $i = 1$ or 2: Vertices in S_1 dominate all the vertices in S_2 which again dominate all the vertices in S_3: There exists exactly one edge from S_3 to S_1 and it goes from the terminal component in $T[S_3]$ to the initial component in $T[S_1]$.
Type 2r. For some $r \geq 1$, there exists a partition S_1, S_2, $\cdots$, S_{2r+2} of $V(T)$ into disjoint non empty subsets with the following conditions. $y_i \in S_1$, $y_{3-i} \in S_2$, $x_{3-i} \in S_{2r+1}$, $x_i \in S_{2r+2}$ for $i = 1$ or 2: All the edges between S_i and S_j where $i < j$ go from S_i to S_j with the following exceptions: There exists precisely one edge from S_j to S_{j-2}, $j = 3, \cdots, 2r + 2$, and it goes from the terminal component in $T[S_j]$ to the initial component in $T[S_{j-2}]$.
Type 2r+1. For some $r \geq 1$, there exists a partition S_1, S_2, $\cdots$, S_{2r+3} of $V(T)$ into disjoint non empty subsets with the following conditions. $y_i \in S_1$, $y_{3-i} \in S_2$, $x_i \in S_{2r+2}$, $x_{3-i} \in S_{2r+3}$ for $i = 1$ or 2: All the edges between S_i and S_j where $i < j$ go from S_i to S_j with the following exceptions: There exists precisely one edge from S_j to S_{j-2}, $j = 3, \cdots, 2r + 3$, and it goes from the terminal component in $T[S_j]$ to the initial component in $T[S_{j-2}]$. □

[Lemma 2][1] *Let T be a tournament and let x_1, y_1, x_2, y_2 be different vertices such that T contains an (x_i, y_i)-path $i = 1, 2$. Then T has edge-disjoint (x_1, y_1)-, (x_2, y_2)-paths unless x_1, y_1, x_2, y_2 all belong to the same component T_j of T and (T, x_1, x_2, y_1, y_2) is of one of the types 1a, 1b, 2r or 2r+1 for some $r \geq 1$, in Definition 1, in which case T does not have these paths.* □

Based on the property and these lemmas, we get the following procedure for examining whether edge-disjoint (x_1, y_1)-, (x_2, y_2)-paths exist or not.

Procedure Check_Existence
begin
(Step 1) Check whether T has an (x_i, y_i)-path for $i = 1$ and 2, not necessary edge-disjoint. If not then T does not have edge-disjoint (x_1, y_1)-, (x_2, y_2)-paths and the procedure stops.
(Step 2)
$d^+_{\max}(v) \leftarrow \max\{d^+(w) \mid (v, w) \in E(T)\}$.
$\mathcal{V}^+{}_{\max}[i] \leftarrow w$.
(Step 3) Set the degree $d^+_{\max}(v_i)$ of v_i into array $\mathcal{D}^+[i], i = 1, \cdots, n$, and sort $\mathcal{D}^+[i]$ in the order of ascending degree. Calculate the value of $\mathcal{T}[i]$, $\mathcal{S}[i]$ and $\mathcal{D}if[i]$.
$\mathcal{T}[0] \leftarrow 0$.
for $i = 1, \cdots, n$
begin
$\mathcal{T}[i] \leftarrow \mathcal{T}[i-1] + \mathcal{D}^+[i]$.
$\mathcal{S}[i] \leftarrow \binom{i}{2}$.
$\mathcal{D}if[i] \leftarrow \mathcal{T}[i] - \mathcal{S}[i]$.
end
(Step 4) Check whether x_1, y_1, x_2 and y_2 all belong to the same component T_j of T. If not then T has edge-disjoint (x_1, y_1)-, (x_2, y_2)-paths and the procedure stops.
(Step 5) Let $T = T_j$ (namely, throw away the rest of T).
(Step 6) Assume that $d^+(x_i) \leq d^+(x_{3-i})$.
(In the following steps, we examine whether T is divided into some component or not by exchanging the direction of an edge (v, w).)
In the order of ascending degree, check Condition 1 below and get a vertex v satisfying the condition first. If there is no vertex satisfying Condition 1, edge-disjoint (x_1, y_1)-, (x_2, y_2)-paths exist and stops.
Condition 1 : at least one of $\mathcal{D}if[1], \cdots, \mathcal{D}if[I_{\max}(d^+_{\max}(v)) - 1]$ has 1 and its index is not less than $I_{\min}(d^+(x_i))$.
On $\mathcal{D}if[I_{\max}(v)], \cdots, \mathcal{D}if[I_{\max}(d^+_{\max}(v)) - 1]$, find index i such that $\mathcal{D}if[i] = 1$. Assume here that an (v, w) is selected and $\mathcal{D}if[i] = \mathcal{D}if[j] = \cdots = \mathcal{D}if[k] = 1$, $i < j < \cdots < k$ hold. By exchanging the direction of the edge (v, w), T is not strong and an induced subgraph $D[\{v_1, v_2, \cdots, v_i\}]$ is a component T_1, $D[\{v_{i+1}, \cdots, v_j\}]$ is $T_2, \cdots, D[\{v_{k+1}, \cdots, v_n\}]$ is T_l.
(Step 7) Find a component including x_i. We here assume that $x_i \in V(T_k)$.
(Case I) When x_{3-i} also exists in T_k.
(I.I) If either y_i or y_{3-i} belongs to $T_1, \cdots, T_k$, edge-disjoint (x_1, y_1)-, (x_2, y_2)-paths exist in T and the procedure stops.
(I.II) If both y_i and y_{3-i} exist in $T_{k+1}, \cdots, T_l$, edge-disjoint (x_1, y_1)-, (x_2, y_2)-paths do not exist in T and the procedure stops.
(Case II) When x_{3-i} exists in T_j, $k < j$.
(II.I) If y_{3-i} exists in $V(T_j)$, edge-disjoint (x_1, y_1)-, (x_2, y_2)-paths exist in T and the procedure stops.
(II.II) If y_{3-i} exists in $V(T'_j)$, $j' < j$, edge-disjoint (x_1, y_1)-, (x_2, y_2)-paths exist in T and the procedure stops.
(II.III) If y_{3-i} exists in $V(T'_j)$, $j' > j$, edge-disjoint (x_1, y_1)-, (x_2, y_2)-paths do not exist in T and the procedure stops.
(Case III) When x_{3-i}, y_i and y_{3-i} all belong to T_l. Let $T = T_l$ and $x_i = w$, namely, remove vertices and edges which do not belong to T_l from T. Actually, it is sufficient for the procedure to changes the value of arrays $\mathcal{D}^+$, $\mathcal{D}if$, $d^+_{\max}$, $\mathcal{V}^+$. Go to Step 6.
end. □

[Lemma 3] *On tournament,* Procedure **Check_Existence** *can examine whether edge-disjoint (x_1, y_1)-, (x_2, y_2)-paths exist or not.* □

[Theorem 1] Procedure **Check_Existence** *can examine the existence of edge-disjoint (x_1, y_1)-, (x_2, y_2)-paths in $O(n^2)$ time.* □

We obtain the following result though we do not write details because of the lack of space.

[Theorem 2] Procedure **Find_Path** *can find edge-disjoint (x_1, y_1)-, (x_2, y_2)-paths in $O(n^2)$ time.* □

References

[1] J. Bang-Jensen: "Edge-Disjoint In- and Out-Branchings in Tournaments and Related Path Problems", *Journal Combinatorial Theory, Series B*, **51**, pp.1-23, 1991.