1997年度日本オペレーションズ・リサーチ学会
秋季研究発表会

Some Mathematical Programming Issues in DEA

01302170 埼玉大学 刀根 薫 TONE Kaoru

1 Introduction

In this paper, we will analyze the equivalence of the original ratio form fractional program for the CCR model and the derived linear program under the semi-positive data set assumption. Then, the uniqueness issues of the solutions will be discussed in detail.

2 Fractional Program with Semipositive Data Set

$$\begin{aligned} (FP_o) \quad \max\theta &= uy_o/vx_o &(1)\\ st. \quad uy_j/vx_j &\le 1\ (j=1,\ldots,n) &(2)\\ v &\ge 0,\ \ u\ge 0. &(3)\end{aligned}$$

$$\begin{aligned} (LP_o) \quad \max\theta &= uy_o &(4)\\ st. \quad vx_o &= 1 &(5)\\ uY &\le vX,\ \ v\ge 0,\ \ u\ge 0. &(6)\end{aligned}$$

$$\begin{aligned} (DLP_o) \quad \min\theta & &(7)\\ st. \quad \theta x_o &= X\lambda + s_x &(8)\\ y_o &= Y\lambda - s_y &(9)\\ \lambda &\ge 0, s_x\ge 0, s_y\ge 0 &(10)\end{aligned}$$

Definition 1 (Semi-positive Data Set) *x_j and y_j $(j=1,\ldots,n)$ are nonnegative and nonzero.*

2.1 When DMU_o has no Slacks

If a DMU_o has an optimal max-slack solution $(\theta=\theta^*, \lambda=\lambda^*, s_x^*=0, s_y^*=0)$, then, by the strong theorem of complementarity, there exists a positive optimal solution (v^*,u^*) to (LP_o) and it holds

$$v^*X \ge u^*Y > 0.$$

Thus, for (v^*,u^*), the ratio form

$$\frac{u^*y_j}{v^*x_j}$$

has a definite value for every DMU. Therefore, for DMU_o, (LP_o) is equivalent to (FP_o).

2.2 When DMU_o has Slacks

If DMU_o has an optimal max-slack solution with $s_x^*\ne 0$ and/or $s_y^*\ne 0$, we replace (FP_o) with

$$\begin{aligned} (\overline{FP}_o) \quad &\max \frac{uy_o}{v_ox_o} &(11)\\ \text{subject to} \quad &\frac{uy_j}{vx_j} \le 1\ (\forall j) &(12)\\ &v\ge \varepsilon e,\ \ u \ge \varepsilon e, &(13)\end{aligned}$$

where e is a row vector with all elements equal to 1 and the symbol ε represents the infinitesimal (small) positive number. Thus, the fractional terms in $(\overline{FP}_o)$ have definite values by the semipositivity assumption on X.

The derived $(\overline{LP}_o)$ and $(\overline{DLP}_o)$ turn out to have the added constraint $v\ge\varepsilon e$, $u\ge\varepsilon e$ and the objective min $\theta-\varepsilon(es_x+es_y)$, respectively. Since ε is infinitesimally small, $(\overline{DLP}_o)$ has the same optimal max-slack solution $(\theta^*,\lambda^*,s_x^*,s_y^*)$ with (DLP_o). For a feasible solution $(\bar{v},\bar{u})$ for $(\overline{LP}_o)$, the optimality conditions are:

$$\text{if } s_{x_j}^*>0, \text{ then } \bar{v}_j=\varepsilon$$

and

$$\text{if } s_{y_j}^*>0, \text{ then } \bar{u}_j=\varepsilon.$$

As ε approaches zero, the $(\bar{v},\bar{u})$ converges to an optimal solution (v^*,u^*) of (LP_o) as its limit.

Thus, it can be concluded that we are solving the following *supremum* (sup) programming problem (SP_o) in the *positive* orthant of (v,u), instead of (FP_o).

$$\begin{aligned} (SP_o) \quad &\sup \frac{uy_o}{v_ox_o} &(14)\\ \text{subject to} \quad &\frac{uy_j}{vx_j} \le 1\ (j=1,\ldots,n) &(15)\\ &v>0,\ \ u > 0. &(16)\end{aligned}$$

3 On the Uniqueness of the Solutions

Phase I Objective $\min\theta$ (17)

Phase II Objective $\max\omega = w_xs_x + w_ys_y$

3.1 General Scheme of LP Computation

The LP problem (DLP'_o) can be formulated as:

$$\text{Phase I objective min } z_1 = cx \quad (18)$$
$$\text{Phase II objective min } z_2 = dx \quad (19)$$
$$\text{subject to } Ax = b \quad (20)$$
$$x \geq 0. \quad (21)$$

Let a submatrix B of A be an optimal basis for Phase II LP and R be the nonbasic part of A.

Phase I objective	θ^*	0	$-\bar{c}$
Phase II objective	$-\omega^*$	0	$-\bar{d}$
	$\bar{b}$	I	$B^{-1}R$

Definition 2 (Degeneracy)
An optimal basis B is called

1. b-nondegenerate *if* $\bar{b} > 0$, *otherwise* b-degenerate,
2. c-nondegenerate *if* $\bar{c} > 0$, *otherwise* c-degenerate, *and*
3. d-nondegenerate *if* $\bar{d}_j > 0$ *for all* $j \in R$ *with* $\bar{c}_j = 0$, *and* d-degenerate *if for some* $j \in R$, $\bar{d}_j = 0$ *and* $\bar{c}_j = 0$.

3.2 The 'c–nondegenerate' Case

In this case, all nonbasic variables have negative simplex criteria in the c-row of the optimal tableau. Thus, x_j $(j \in R)$ cannot have a positive value in every optimal solution and hence $((\boldsymbol{x}^B)^* = \bar{\boldsymbol{b}}, (\boldsymbol{x}^R)^* = \boldsymbol{0})$ is the only optimal solution. In CCR terminology, the optimal solution $(\theta^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$ is unique. We will now discuss the b-degeneracy issue in this case.

(i) When the basis B is b-nondegenerate.
In this case, the basis B is the only optimal basis and both (DLP_o) and (LP_o) have unique optimal solutions.

(ii) When the basis B is b-degenerate.
In this case, we have only one optimal solution $(\boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$ for (DLP_o). However, the optimal solution $(\boldsymbol{v}^*, \boldsymbol{u}^*)$ is not necessarily unique.

3.3 The 'c–degenerate but d–nondegenerate' Case

In this case, the optimal basic solution $(\boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$ corresponds to the unique vertex that maximizes $\omega = \boldsymbol{e}\boldsymbol{s}_x + \boldsymbol{e}\boldsymbol{s}_y$ in the (DLP_o) feasible region with $\theta = \theta^*$. Thus, the solution is unique. However, this uniqueness depends on the objective function form of Phase II.

3.4 The 'c–degenerate and d–degenerate' Case

In this case, the optimal solution $(\boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$ is seemingly not unique. However, we should be careful in deciding the existence of substantially multiple optimal solutions. For this purpose, we consider a Phase III LP, based on an optimal basis B and its optimal solution $(\boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$ for Phase II, as follows.

We maximize the objective function :

$$\eta = \sum_j \lambda_j \quad (j \in R \text{ with } \bar{c}_j = 0 \text{ and } \bar{d}_j = 0)$$
$$+ \sum_j \lambda_j \quad (j \in B \text{ with } \lambda_j^* = 0), \qquad (22)$$

subject to the constraints of (DLP_o), added by $\theta = \theta^*$ and $\boldsymbol{e}\boldsymbol{s}_x + \boldsymbol{e}\boldsymbol{s}_y = \boldsymbol{e}\boldsymbol{s}_x^* + \boldsymbol{e}\boldsymbol{s}_y^*$.

Let the optimal value of η be η^*. Then, if $\eta^* > 0$, it is found that (DLP_o) has multiple optimal solutions. On the other hand, if $\eta^* = 0$, then $(\boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$ is the only one solution of (DLP_o).

3.5 Summary of the Degeneracy and Uniqueness Issues

Uniqueness and degeneracy are summarized in Table 1 and Table 2, where in Table 2, 'not unique' means 'not necessarily unique'.

Table 1: Degeneracy and Uniqueness in (DLP_o)

	c-nond.	c-deg.		
		d-nond.	d-deg.	
			$\eta^* = 0$	$\eta^* > 0$
$(\boldsymbol{\lambda}^*, \boldsymbol{s}^*)$	unique	unique	unique	not unique

Table 2: Degeneracy and Uniqueness in (LP_o)

	b-nondegenerate	b-degenerate
$(\boldsymbol{v}^*, \boldsymbol{u}^*)$	unique	not unique

Reference [1] Charnes, Cooper and Thrall, JPA, 2, 197-237, 1991.