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Further Results for Multiclass M/G/1 Queues with Feedback (II)

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4 Initial sojourn times.

Let us consider the e^{th} customer arrived at station j with the nonpreemptive FCFS discipline at σ_l^e and he belongs to class β ($e = 1, 2, \ldots$ and $l = 0, 1, \ldots$). Let $\mathbf{Y}(\sigma_l^e) = \mathbf{Y} = (j, \beta, \iota, a, r, \mathbf{V}, \mathbf{n}) \in \mathcal{E}$ be the state of the system at his arrival epoch. Since $W_{i\alpha}^I(\mathbf{Y}, e, l) = 0$ and $G_{i\alpha}^I(\mathbf{Y}, e, l) = 0$ for $(i, \alpha) \neq (j, \beta)$, we consider the case where $(i, \alpha) = (j, \beta) \in \mathcal{S}$. Let $J_1 \equiv J + J_c$.

The customer set $\mathcal{C}_j^F$ is composed of a customer being served and customers in one of groups $1, \ldots, j$ who are in the system at time σ_l^e (except for the arriving customer). The initial sojourn time of the customer is composed of a group $j-1$ busy period initiated with $\{\mathbf{Y}; \mathcal{C}_j^F\}$, and the e^{th} customer's service $S_{j\beta}$. Hence,

$$W_{j\beta}^I(\mathbf{Y}, e, l) = E[B^{j-1}(\mathbf{Y}; \mathcal{C}_j^F) + S_{j\beta}], \quad (4.1)$$

$$G_{j\beta}^I(\mathbf{Y}, e, l) = E[S_{j\beta}]E[B^{j-1}(\mathbf{Y}; \mathcal{C}_j^F)] + \overline{s^2}_{j\beta}/2. \quad (4.2)$$

By appropriately choosing nonnegative vectors $\mathbf{w}^{j\beta}$, $\mathbf{g}^{j\beta} \in \mathcal{R}_+^{J_1}$ and nonnegative constants $\varphi^{j\beta}(\iota, a)$, $w^{j\beta}(\iota, a)$, $\eta^{j\beta}(\iota, a)$, $g^{j\beta}(\iota, a) \in \mathcal{R}_+$ ($(j, \beta) \in \mathcal{S}, (\iota, a) \in \mathcal{S}_0$), we may simply express

$$W_{j\beta}^I(\mathbf{Y}, e, l) = r\varphi^{j\beta}(\iota, a) + (\mathbf{v}, \mathbf{n})\mathbf{w}^{j\beta} + w^{j\beta}(\iota, a). \quad (4.3)$$

$$G_{j\beta}^I(\mathbf{Y}, e, l) = r\eta^{j\beta}(\iota, a) + (\mathbf{v}, \mathbf{n})\mathbf{g}^{j\beta} + g^{j\beta}(\iota, a), \quad (4.4)$$

for $\mathbf{Y} = (j, \beta, \iota, a, r, \mathbf{V}, \mathbf{n}) \in \mathcal{E}$ and $l = 0, 1, \ldots$.

Now we would like to consider the work at each station and the number of customers in each class at each station at the completion epoch of the initial stay of the e^{th} customer. Then

$$E[v_k(\sigma_{l+1}^e) | \mathbf{Y}(\sigma_l^e) = \mathbf{Y}] = \sum_{\gamma=1}^{L_k} \{\overline{V}_{k\gamma}^{j-1}(\mathbf{Y}; \mathcal{C}_j^F) + \lambda_{k\gamma} E[S_{j\beta}] E[S_{k\gamma}]\}, \quad (4.5)$$

$$E[n_{k\gamma}(\sigma_{l+1}^e) | \mathbf{Y}(\sigma_l^e) = \mathbf{Y}] = \overline{N}_{k\gamma}^{j-1}(\mathbf{Y}; \mathcal{C}_j^F) + \lambda_{k\gamma} E[S_{j\beta}]. \quad (4.6)$$

for $(k, \gamma) \in \mathcal{S}$ ($\overline{V}_{k\gamma}^{j-1}(\mathbf{Y}; \mathcal{C}_j^F) \equiv 0, \overline{N}_{k\gamma}^{j-1}(\mathbf{Y}; \mathcal{C}_j^F) \equiv 0$ if $k < j$). By appropriately choosing a nonnegative matrix $\mathbf{U}^{j\beta} \in \mathcal{R}_+^{J_1 \times J_1}$ and nonnegative vectors $\boldsymbol{v}^{j\beta}(\iota, a), \mathbf{u}^{j\beta}(\iota, a) \in \mathcal{R}_+^{1 \times J_1}$ ($(j, \beta) \in \mathcal{S}, (\iota, a) \in \mathcal{S}_0$), we can simply express as follows:

$$E[(\mathbf{v}(\sigma_{l+1}^e), \mathbf{n}(\sigma_{l+1}^e)) | \mathbf{Y}(\sigma_l^e) = \mathbf{Y}] = r\boldsymbol{v}^{j\beta}(\iota, a) + (\mathbf{v}, \mathbf{n})\mathbf{U}^{j\beta} + \mathbf{u}^{j\beta}(\iota, a), \quad (4.7)$$

for $\mathbf{Y} = (j, \beta, \iota, a, r, \mathbf{V}, \mathbf{n}) \in \mathcal{E}$ and $l = 0, 1, \ldots$.

5 Expressions of the cost functions.

In this section and the next section, we derive the explicit formulae of the cost functions $W_{i\alpha}(\cdot)$ and $G_{i\alpha}(\cdot)$ under some assumptions. Let $J_2 \equiv (J + J_c) \times J_c$, and define:

$$\boldsymbol{\phi}_0^{i\alpha} \equiv \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{w}^{i\alpha} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \sim (1,1) \\ \\ \\ \sim (i,\alpha) \\ \\ \\ \sim (J, L_J) \end{matrix} \in \mathcal{R}^{J_2 \times 1}, \quad \boldsymbol{\eta}_0^{i\alpha} \equiv \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{g}^{i\alpha} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathcal{R}^{J_2 \times 1},$$

$$\mathbf{UQ} = \begin{pmatrix} \mathbf{U}^{11} p_{11,11} & \cdots & \mathbf{U}^{11} p_{11,JL_J} \\ \vdots & \ddots & \vdots \\ \mathbf{U}^{JL_J} p_{JL_J,11} & \cdots & \mathbf{U}^{JL_J} p_{JL_J,JL_J} \end{pmatrix} \in \mathcal{R}^{J_2 \times J_2}.$$

We now suppose $(\mathbf{I} - \mathbf{UQ})^{-1}$ exists, where $\mathbf{I}$ is an identity matrix in $\mathcal{R}^{J_2 \times J_2}$. Then we can define

$$\begin{pmatrix} \mathbf{w}_{i\alpha}(1,1) \\ \vdots \\ \mathbf{w}_{i\alpha}(J, L_J) \end{pmatrix} \equiv (\mathbf{I} - \mathbf{UQ})^{-1} \boldsymbol{\phi}_0^{i\alpha} \in \mathcal{R}^{J_2 \times 1}, \quad (5.1)$$

$(\mathbf{g}_{i\alpha}(1,1), \ldots, \mathbf{g}_{i\alpha}(J, L_J))' \in \mathcal{R}^{J_2 \times 1}$ is defined in the same mannar as the above expression for $(\mathbf{w}_{i\alpha}(1,1), \ldots, \mathbf{w}_{i\alpha}(J, L_J))'$ by substituting $\boldsymbol{\eta}_0^{i\alpha}$ for $\boldsymbol{\phi}_0^{i\alpha}$. Further we define $\boldsymbol{\phi}_1^{i\alpha} \in \mathcal{R}^{J_c \times 1}$ such that

$$\boldsymbol{\phi}_1^{i\alpha} \equiv \begin{pmatrix} \mathbf{u}^{11}(0,0) \sum_{k=1}^{J} \sum_{\gamma=1}^{L_k} p_{11,k\gamma} \mathbf{w}_{i\alpha}(k,\gamma) \\ \vdots \\ \mathbf{u}^{i'\alpha'}(0,0) \sum_{k=1}^{J} \sum_{\gamma=1}^{L_k} p_{i'\alpha',k\gamma} \mathbf{w}_{i\alpha}(k,\gamma) \\ w^{i\alpha}(0,0) + \mathbf{u}^{i\alpha}(0,0) \sum_{k=1}^{J} \sum_{\gamma=1}^{L_k} p_{i\alpha,k\gamma} \mathbf{w}_{i\alpha}(k,\gamma) \\ \mathbf{u}^{i''\alpha''}(0,0) \sum_{k=1}^{J} \sum_{\gamma=1}^{L_k} p_{i''\alpha'',k\gamma} \mathbf{w}_{i\alpha}(k,\gamma) \\ \vdots \\ \mathbf{u}^{JL_J}(0,0) \sum_{k=1}^{J} \sum_{\gamma=1}^{L_k} p_{JL_J,k\gamma} \mathbf{w}_{i\alpha}(k,\gamma) \end{pmatrix},$$

where $(i', \alpha') = (i, \alpha - 1)$ for $\alpha \neq 1$, or $(i', \alpha') = (i-1, L_{i-1})$ for $\alpha = 1$, and where $(i'', \alpha'') = (i, \alpha + 1)$ for $\alpha \neq L_i$, or $(i'', \alpha'') = (i+1, 1)$ for $\alpha = L_i$. $\boldsymbol{\eta}_1^{i\alpha} \in \mathcal{R}^{J_c \times 1}$ is defined in the same mannar as the above expression by substituting $\mathbf{g}_{i\alpha}(k, \gamma)$ and $g^{i\alpha}(0,0)$ for $\mathbf{w}_{i\alpha}(k, \gamma)$ and $w^{i\alpha}(0,0)$, respectively. From **Assumption 1**, $(\mathbf{I} - \mathbf{P})^{-1}$ exists. Then we can define

$$\begin{pmatrix} w_{i\alpha}(1,1) \\ \vdots \\ w_{i\alpha}(J, L_J) \end{pmatrix} \equiv (\mathbf{I} - \mathbf{P})^{-1} \boldsymbol{\phi}_1^{i\alpha} \in \mathcal{R}^{J_c \times 1}. \quad (5.2)$$

$(g_{i\alpha}(1,1),\ldots,g_{i\alpha}(J,L_J))' \in \mathcal{R}^{J_c\times 1}$ is defined in the same mannar as the above expressions for $(w_{i\alpha}(1,1),\ldots,w_{i\alpha}(J,L_J))'$ by substituting $\boldsymbol{\eta}_1^{i\alpha}$ for $\boldsymbol{\phi}_1^{i\alpha}$. Further we define

$$w_{i\alpha}(j,\beta,\iota,a) \equiv \begin{cases} w^{i\alpha}(\iota,a)+\sum_{k=1}^{J}\sum_{\gamma=1}^{L_k} p_{i\alpha,k\gamma}\{\mathbf{u}^{i\alpha}(\iota,a)\mathbf{w}_{i\alpha}(k,\gamma) \\ \quad +w_{i\alpha}(k,\gamma)\}, \qquad (j,\beta)=(i,\alpha), \\ \sum_{k=1}^{J}\sum_{\gamma=1}^{L_k} p_{j\beta,k\gamma}\{\mathbf{u}^{j\beta}(\iota,a)\mathbf{w}_{i\alpha}(k,\gamma)+w_{i\alpha}(k,\gamma)\}, \\ \quad (j,\beta)\neq(i,\alpha), \end{cases} \tag{5.3}$$

$$\varphi_{i\alpha}(j,\beta,\iota,a) \equiv \begin{cases} \varphi^{i\alpha}(\iota,a)+\boldsymbol{v}^{i\alpha}(\iota,a)\sum_{k=1}^{J}\sum_{\gamma=1}^{L_k} p_{i\alpha,k\gamma}\mathbf{w}_{i\alpha}(k,\gamma), \\ \quad (j,\beta)=(i,\alpha), \\ \boldsymbol{v}^{j\beta}(\iota,a)\sum_{k=1}^{J}\sum_{\gamma=1}^{L_k} p_{j\beta,k\gamma}\mathbf{w}_{i\alpha}(k,\gamma), \quad (j,\beta)\neq(i,\alpha), \end{cases} \tag{5.4}$$

for $(i,\alpha),(j,\beta)\in\mathcal{S},(\iota,a)\in\mathcal{S}_0$. $g_{i\alpha}(j,\beta,\iota,a)$ and $\eta_{i\alpha}(j,\beta,\iota,a)$ is defined in the same mannar as the above expressions for $w_{i\alpha}(j,\beta,\iota,a)$ and $\varphi_{i\alpha}(j,\beta,\iota,a)$, respectively, by substituting $g^{i\alpha}(\iota,a)$, $\mathbf{g}_{i\alpha}(k,\gamma)$, $g_{i\alpha}(k,\gamma)$ and $\eta^{i\alpha}(\iota,a)$ for $w^{i\alpha}(\iota,a)$, $\mathbf{w}_{i\alpha}(k,\gamma)$, $w_{i\alpha}(k,\gamma)$ and $\varphi^{i\alpha}(\iota,a)$.

For any $(i,\alpha)\in\mathcal{S}$ and $e=1,2,\ldots$, we define

$$\hat{W}_{i\alpha}(\mathbf{Y},e,l) \equiv \begin{cases} r\varphi_{i\alpha}(j,\beta,\iota,a)+(\mathbf{v},\mathbf{n})\mathbf{w}_{i\alpha}(j,\beta) \\ \quad +w_{i\alpha}(j,\beta,\iota,a), \qquad l=0, \\ (\mathbf{v},\mathbf{n})\mathbf{w}_{i\alpha}(j,\beta)+w_{i\alpha}(j,\beta), \qquad l>0, \end{cases} \tag{5.5}$$

$$\hat{G}_{i\alpha}(\mathbf{Y},e,l) \equiv \begin{cases} r\eta_{i\alpha}(j,\beta,\iota,a)+(\mathbf{v},\mathbf{n})\mathbf{g}_{i\alpha}(j,\beta) \\ \quad +g_{i\alpha}(j,\beta,\iota,a), \qquad l=0, \\ (\mathbf{v},\mathbf{n})\mathbf{g}_{i\alpha}(j,\beta)+g_{i\alpha}(j,\beta), \qquad l>0, \end{cases} \tag{5.6}$$

where $\mathbf{Y}=(j,\beta,\iota,a,r,\mathbf{V},\mathbf{n})\in\mathcal{E}$,

Now we make the following assumption.

Assumption 3. $(\mathbf{UQ})^m \to \mathbf{O}$ as $m\to\infty$. □

Under these assumptions, we can show:

- $\hat{W}_{i\alpha}(\mathbf{Y},e,l)$ defined in (5.5) and $\hat{G}_{i\alpha}(\mathbf{Y},e,l)$ defined in (5.6) are solutions of equations (2.7) and (2.11), respectively. □

6 Steady state values.

The throughputs $\{\dot{\theta}_{i\alpha}:(i,\alpha)\in\mathcal{S}\}$ are defined by

$$\dot{\theta}_{i\alpha}=\lambda_{i\alpha}+\textstyle\sum_{j=1}^{J}\sum_{\beta=1}^{L_j}\dot{\theta}_{j\beta}p_{j\beta,i\alpha}, \quad (i,\alpha)\in\mathcal{S}. \tag{6.1}$$

Then we define

$$\check{r}^{\iota a} \equiv \begin{cases} \dot{\theta}_{\iota a}\overline{s^2}_{\iota a}/2, & (\iota,a)\in\mathcal{S}, \\ 0, & (\iota,a)=(0,0), \end{cases} \tag{6.2}$$

$$\check{q}^{\iota a} \equiv \begin{cases} \dot{\theta}_{\iota a}E[S_{\iota a}], & (\iota,a)\in\mathcal{S}, \\ 1-\rho_J^{\dagger}, & (\iota,a)=(0,0), \end{cases} \tag{6.3}$$

$$\check{\mathbf{s}} \equiv (\sum_{a=1}^{L_1}\check{r}^{1a},\ldots,\sum_{a=1}^{L_J}\check{r}^{Ja},\check{q}^{11},\ldots,\check{q}^{JL_J}). \tag{6.4}$$

Now we define the following customer average values:

$$\bar{W}_{i\alpha}(\cdot) \equiv \lim_{N\to\infty}\frac{1}{N}\sum_{e=1}^{N}W_{i\alpha}^{e}, \quad (i,\alpha)\in\mathcal{S}. \tag{6.5}$$

$$\bar{G}_{i\alpha}(\cdot) \equiv \lim_{N\to\infty}\frac{1}{N}\sum_{e=1}^{N}G_{i\alpha}^{e}, \quad (i,\alpha)\in\mathcal{S}, \tag{6.6}$$

if these limits shall exist. The time average values of components of states are defined by:

$$\begin{aligned} \tilde{n}_{i\alpha} &\equiv \lim_{t\to\infty}\textstyle\int_0^t n_{i\alpha}(s)ds/t, \quad (i,\alpha)\in\mathcal{S}, \\ \tilde{v}_i &\equiv \lim_{t\to\infty}\textstyle\int_0^t v_i(s)ds/t, \quad i=1,\ldots,J, \\ (\tilde{\mathbf{v}},\tilde{\mathbf{n}}) &\equiv (\tilde{v}_1,\ldots,\tilde{v}_J,\tilde{n}_{11},\ldots,\tilde{n}_{JL_J}). \end{aligned} \tag{6.7}$$

Under some assumptions, it can be shown

$$\bar{W}_{i\alpha}(\cdot) = \sum_{j=1}^{J}\sum_{\beta=1}^{L_j}\frac{\lambda_{j\beta}}{\lambda}\{\tilde{w}_{i\alpha}(j,\beta)+(\tilde{\mathbf{v}},\tilde{\mathbf{n}})\mathbf{w}_{i\alpha}(j,\beta)\}, \tag{6.8}$$

$$\bar{G}_{i\alpha}(\cdot) = \sum_{j=1}^{J}\sum_{\beta=1}^{L_j}\frac{\lambda_{j\beta}}{\lambda}\{\tilde{g}_{i\alpha}(j,\beta)+(\tilde{\mathbf{v}},\tilde{\mathbf{n}})\mathbf{g}_{i\alpha}(j,\beta)\}, \tag{6.9}$$

$$\tilde{w}_{i\alpha}(j,\beta) \equiv \sum\sum_{(\iota,a)\in\mathcal{S}_0}\{\check{r}^{\iota a}\varphi_{i\alpha}(j,\beta,\iota,a)+\check{q}^{\iota a}w_{i\alpha}(j,\beta,\iota,a)\}.$$

$$\tilde{g}_{i\alpha}(j,\beta) \equiv \sum\sum_{(\iota,a)\in\mathcal{S}_0}\{\check{r}^{\iota a}\eta_{i\alpha}(j,\beta,\iota,a)+\check{q}^{\iota a}g_{i\alpha}(j,\beta,\iota,a)\}.$$

for $(i,\alpha),(j,\beta)\in\mathcal{S}$.

By the generalized Little's formula ($H=\lambda G$: [1], [2]),

$$(\tilde{\mathbf{v}},\tilde{\mathbf{n}})+\check{\mathbf{s}}=(\tilde{\mathbf{v}},\tilde{\mathbf{n}})\mathbf{S}+\mathbf{s}, \tag{6.10}$$

or

$$(\tilde{\mathbf{v}},\tilde{\mathbf{n}})=(\mathbf{s}-\check{\mathbf{s}})(\mathbf{I}-\mathbf{S})^{-1}, \tag{6.11}$$

where

$$\begin{aligned} \mathbf{S} \equiv& \sum_{j=1}^{J}\sum_{\beta=1}^{L_j}\lambda_{j\beta}(\sum_{\alpha=1}^{L_1}\mathbf{g}_{1\alpha}(j,\beta),\ldots,\sum_{\alpha=1}^{L_J}\mathbf{g}_{J\alpha}(j,\beta), \\ &\mathbf{w}_{11}(j,\beta),\ldots,\mathbf{w}_{JL_J}(j,\beta))\in\mathcal{R}^{J_1\times J_1}, \\ \mathbf{s} \equiv& \sum_{j=1}^{J}\sum_{\beta=1}^{L_j}\lambda_{j\beta}(\sum_{\alpha=1}^{L_1}\tilde{g}_{1\alpha}(j,\beta),\ldots,\sum_{\alpha=1}^{L_J}\tilde{g}_{J\alpha}(j,\beta), \\ &\tilde{w}_{11}(j,\beta),\ldots,\tilde{w}_{JL_J}(j,\beta))\in\mathcal{R}^{1\times J_1}. \end{aligned}$$

Finally let $\bar{W}_{i\alpha}(j,\beta)$ be the steady state value of the sojourn time of a customer spend at station i as a class α customer, given that the customer arrives at station j as a class β from outside the system ($(i,\alpha),(j,\beta)\in\mathcal{S}$). Then we have

$$\bar{W}_{i\alpha}(j,\beta)=(\mathbf{s}-\check{\mathbf{s}})(\mathbf{I}-\mathbf{S})^{-1}\mathbf{w}_{i\alpha}(j,\beta)+\tilde{w}_{i\alpha}(j,\beta), \tag{6.12}$$

for $(i,\alpha),(j,\beta)\in\mathcal{S}$. □

References

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