The effect of both sides operational flexibility in capacity investment under model uncertainty*

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1. Introduction

This paper analyzes a firm's capacity investment problem under uncertainty. The firm faces uncertainty about the output price and can not uniquely identify its distribution. The firm treats the price dynamics as an approximation of its actual dynamics. Then, the firm decides its managerial strategy under model uncertainty. To deal with the model uncertainty, we employ the robust control approach developed by Hansen, Sargent, and their coauthors (Hansen and Sargent, 2001; Hansen et al., 2006).

The firm considers changing its capacity in response to the output demand. The firm expands (resp., contracts) the capacity if the demand is sufficiently large (resp., low). Consequently, the firm has two types of operational flexibility. We reveal the effect of both sides' operational flexibility on the firm's decision-making under model uncertainty.

2. Firm's Problem

Suppose that a firm produces an output Q_t in a competitive market and receives the operating profit $\pi(P_t, Q_t) = (P_t - c)Q_t$, where P_t is the output price and $c > 0$ is the operating cost.

The firm changes the level of its capacity depending on the demand. The inverse demand function, i.e., the output price, is exogenously given. We assume that the firm produces one unit output per unit capacity. This means that the change in Q is equivalent to the change in the capacity.

The firm has two types of managerial flexibility. One is the flexibility to expand the capacity from Q_0 to $Q_H := Q_0 + \Delta Q$ with the constant $\Delta Q \in (0, Q_0)$ for $P_t \geq p_H$. The other one is to contract the capacity to $Q_L := Q_0 - \Delta Q$ for $P_t \leq p_L$. Expanding the capacity by ΔQ costs $q_H \Delta Q$, while reducing the capacity by ΔQ yields sales gains of $q_L \Delta Q$. Here, q_H is the purchasing price of capacity and q_L is the selling price of capacity with $q_L < q_H$.

The dynamics of the output price P_t is governed by the following:

$$\mathrm{d}P_t = \mu P_t \mathrm{d}t + \sigma P_t \mathrm{d}W_t, \quad P_0 = p > 0, \qquad (1)$$

where $\mu, \sigma > 0$ and W_t is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$, where $\mathcal{F}_t$ is generated by W_t.

The firm's manager feels the possibility of errors in identifying the reference probabilities and treats (1) as an approximation. That is, the firm's manager faces model uncertainty. To express the model uncertainty, we introduce a set of equivalent probability measures, $\mathcal{P}$, on $(\Omega, \mathcal{F})$. Then, the reference probability measure $\mathbb{P}$ is replaced by another equivalent probability measure $\mathbb{Q} \in \mathcal{P}$.

We replace W_t in (1) by $W_t^{\mathbb{Q}} + \int_0^t h_s \mathrm{d}s$, where h_t is progressively measurable and $W_t^{\mathbb{Q}}$ is a Brownian motion under the measure $\mathbb{Q}$. Then, equation (1) is rewritten as:

$$\mathrm{d}P_t = (\mu + \sigma h_t) P_t \mathrm{d}t + \sigma P_t \mathrm{d}W_t^{\mathbb{Q}}, \quad P_0 = p > 0.$$

We use the relative entropy to measure the distance between two probability measures and introduce the discounted relative entropy as in Hansen and Sargent (2001):

$$R(\mathbb{Q}) = \mathbb{E}_{\mathbb{Q}} \left[\int_0^\infty \mathrm{e}^{-rt} \frac{h_t^2}{2} \mathrm{d}t \right],$$

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where r is the discount rate.

The firm's problem is to maximize the net present value of profit under model uncertainty. To this end, the firm optimally chooses the timing τ to expand/contract the capacity even for the worst possible distortion h. The capacity changing time is given by $\tau := \min\{\tau_H, \tau_L\}$, where $\tau_H := \inf\{t \geq 0; P_t \geq p_H\}$ and $\tau_L := \inf\{t \geq 0; P_t \leq p_L\}$. Then, the firm's problem is formulated as:

$$V(p) = \sup_{\tau\in\mathcal{T}} \inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[\int_0^\infty \mathrm{e}^{-rt}\pi(P_t, Q_t)\mathrm{d}t + \theta R(\mathbb{Q}) - \mathrm{e}^{-r\tau}\left(q_H\Delta Q\mathbf{1}_{\{\tau=\tau_H\}} - q_L\Delta Q\mathbf{1}_{\{\tau=\tau_L\}}\right)\right], \quad (2)$$

where V is the value function of the firm's problem and θ is the robustness parameter. $\mathbf{1}_S$ is the indicator function. The second term of the right-hand side, $\theta R(\mathbb{Q})$, represents the cost of taking the distortion.

3. Variational Inequalities of the Firm's Problem

The firm's problem (2) is solved via the variational inequalities:

$$\max\left\{\inf_h\left\{\mathcal{L}V(p) + \pi(p, Q_0) + \theta\frac{h^2}{2}\right\}, \inf_h\{(G_H(p) - q_H\Delta Q) - V(p)\}, \inf_h\{(G_L(p) + q_L\Delta Q) - V(p)\}\right\} = 0, \quad (3)$$

where $\mathcal{L}$ is the infinitesimal operator: $\frac{1}{2}\sigma^2p^2V''(p) + (\mu + \sigma h)pV'(p) - rV(p)$, and G_i $(i = \{0, H, L\})$ the value of operating profit with the cost of taking the distortion:

$$G_i(P_t) := \mathbb{E}_{\mathbb{Q}}\left[\int_t^\infty \mathrm{e}^{-r(t-s)}\pi(P_s; Q_i)\mathrm{d}s + \theta R(\mathbb{Q})|\mathcal{F}_t\right] = \frac{P_tQ_i}{r - (\mu + \sigma h_t)} - \frac{cQ_i}{r} + \theta\frac{h_t^2}{2r}. \quad (4)$$

For the output price is in the continuation region $\mathcal{C} := \{p < p_H\} \cup \{p > p_L\}$, $p \in \mathcal{C}$, the variational inequalities (3) leads to

$$\inf_h\left\{\mathcal{L}V(p) + \pi(p, Q_0) + \theta\frac{h^2}{2}\right\} = 0. \quad (5)$$

Equation (5) yields the the optimal distortion h^* as:

$$h^* = -\frac{\sigma pV'(p)}{\theta}.$$

A general solution to (5) is

$$V(p) = A_1p^{\beta_1} + A_2p^{\beta_2} + \frac{pQ_0}{r - (\mu + \sigma h^*)} - \frac{cQ_0}{r} + \theta\frac{(h^*)^2}{r},$$

where $A_1, A_2 > 0$ are the constants to be determined and $\beta_1 > 1$, $\beta_2 < 0$ are the roots of the particular equation.

If h^* is calculated, four unknown parameters A_1, A_2, p_H, and p_L, are derived by the following value-matching and smooth-pasting conditions:

$$V(p_H; h^*) = G_H(p_H; h_H^*) - q_H\Delta Q,$$
$$V(p_L; h^*) = G_L(p_L; h_L^*) + q_L\Delta Q,$$
$$V'(p_H; h^*) = G'_H(p_H; h_H^*),$$
$$V'(p_L; h^*) = G'_L(p_L; h_L^*),$$

where h_i^* $(i = \{H, L\})$ is derived from the problem when either of two options is exercised:

$$\inf_h\{G_H(p_H) - q_H\Delta Q\} \text{ or } \inf_h\{G_L(p_L) + q_L\Delta Q\}. \quad (6)$$

From (4) and (6), h_i^* is the real solution to the following cubic equation:

$$\frac{\theta\sigma^2}{r}h^3 - \frac{2\theta(r-\mu)}{r}h^2 + \frac{\theta(r-\mu)^2}{r}h + \sigma pQ_i = 0.$$

Notice also that we have $h_t = h^* = h_i^*$ for $P_t = p_i$ $(i = \{H, L\})$.

4. Numerical Analysis

We will present the results of numerical analysis at the conference.

参考文献

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