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Model uncertainty for statistical arbitrage

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1. Introduction

This study presents an optimal boundary for the optimal stopping problem that accounts for model uncertainty. *Model uncertainty* is the uncertainty affecting model assumptions, e.g., the assumed form of the probability distribution and the parameters embedded in the probability distribution. We can use this boundary to make a statistical arbitrage strategy more robust.

Our contribution in this study is to present the most diverted point from the mean-reversion point by solving the optimal stopping problem in a *finite* time horizon, taking into account *model uncertainty*.

More precisely, we utilize the method of [2], who discuss it in the context of convex risk measures. Specifically, we replace the estimated model with a model similar to but different from the estimated model. If we are sure about the estimation, then we adopt a model that is very similar to the estimated model. If not, then we adopt a very different scenario incorporating the worst-case scenario, thus reducing profit. This policy may make the trading code robust. In this formulation, we use Kullback–Leibler divergence to describe the similarity between models and incorporate the minimizing principle related to robustness.

In addition to [2], inspired by [1], [3], and [4], we developed a technique for obtaining the explicit form of the solution to the boundary implying the maximum expected value of the portfolio, which can be used as a benchmark to take a position. Furthermore, we show that the implication of this type of problem is consistent with the certainty equivalent, which leads to the risk premium in the context of expected utility with risk. This property might induce a clear connection between the contexts of risk and uncertainty.

2. Main results

Let us consider a complete, filtered probability space, $(\Omega, \mathcal{F}, P)$, $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$, and on it a bounded, adapted process, X, that satisfies certain regularity conditions. For this process, we consider the following optimal stopping problem:

$$V^\lambda(t,x) \triangleq \operatorname{ess\,sup}_{\tau\in\mathcal{S}_{t,T}} \operatorname{ess\,inf}_{Q\in\mathcal{Q}_t} \left\{\mathbb{E}^Q[Y_\tau|\mathcal{F}_t] + \frac{1}{\lambda}\mathbb{E}^Q\left[\log\left(\frac{dQ}{dP}\right)\Big|\mathcal{F}_t\right]\right\},\ P-a.s. \quad (1)$$

Here, $\mathcal{S}_{t,T}$ is the set of stopping times τ satisfying $t \leq \tau \leq T$, $P-a.s.$; $\mathcal{Q}_t$ is the collection of probability measures Q that are equivalent to P on $\mathcal{F}$, equal to P on $\mathcal{F}_t$, and satisfy a certain integrability condition; Y is a discounted process of X with the discount factor $\rho > 0$, i.e., $Y_t = e^{-\rho t}X_t$ for $0 \leq t \leq T$; $\lambda > 0$ is a constant; and dQ/dP is the density of $Q \in \mathcal{Q}_t$ with respect to P. That is, the density process $\mathcal{Z}^Q$ of Q is the stochastic exponential of a predictable process, $\int_0^\cdot \theta_s^Q dB_s$, such that

$$\mathcal{Z}_t^Q = \exp\left\{\int_0^t \theta_s^Q dB_s - \frac{1}{2}\int_0^t |\theta_s^Q|^2 ds\right\},\ 0 \leq t \leq T,$$

and $\mathcal{Z}_T^Q = dQ/dP$.

Our main aim is to find the explicit solution to (1) by specifying X as an Ornstein-Uhlenbeck (OU) process, such that

$$dX_t = \alpha(\mu - X_t)dt + \sigma dB_t, \quad (2)$$

where $\alpha, \sigma > 0$, $\mu \in \mathbb{R}$ and B is a Brownian motion defined on probability space $(\Omega, \mathcal{F}, P)$.

Theorem 2.1. *Let X be given by* (2)*. Then, the solution to* (1) *is given by*

$$V^\lambda(t,x) = -\frac{1}{\lambda}\log\left(e^{-\lambda e^{-\rho(T-t)}m_x(T-t,x) + \frac{\lambda^2 e^{-2\rho(T-t)}}{2}\sigma_x(T-t)^2} - \int_0^{T-t}\mathcal{K}^x(u,x,b^x(t+u))du\right), \quad (3)$$

where we define

$$m_x(u,x) \triangleq xe^{-\alpha u} + \mu(1-e^{-\alpha u}),\ \sigma_x(u) \triangleq \sqrt{\frac{\sigma^2}{2\alpha}(1-e^{-2\alpha u})}$$

which are the mean and volatility, respectively, of $X^{t,x} = X$ conditioned on $X_t = x$. We also define the kernel

$$\mathcal{K}^x(u,x,b) \triangleq \mathbb{E}_t\left[\mu^x\left(u, X_u^{t,x}\right) I(X_u^{t,x} \geq b)\right],$$

where $\mu^x(u,y) \triangleq -\lambda\left(\alpha e^{-\rho u}\mu - \frac{\lambda}{2}\sigma^2 e^{-2\rho u} - (\alpha+\rho)e^{-\rho u}y\right)\exp\left(-\lambda e^{-\rho u}y\right)$, *the operator* $\mathbb{E}_t[\cdot]$ *is* $\mathbb{E}[\cdot|\mathcal{F}_t]$, *and* $b^x(t)$ *is given by the following equation:*

$$b^x(t) = \log\Bigg(e^{-\lambda e^{-\rho(T-t)}m_x(T-t,b^x(t))+\frac{\lambda^2 e^{-2\rho(T-t)}}{2}\sigma_x(T-t)^2} - \int_0^{T-t}\mathcal{K}^x(u,b^x(t),b^x(t+u))du\Bigg)^{-1/\lambda}, \quad (4)$$

and

$$b^x(T) = \frac{\alpha\mu - \frac{\lambda}{2}\sigma^2}{\alpha+\rho}.$$

3. Implications

The main results indicate the most divergent point. Since $V^\lambda(t,x)$ of (1) is essentially the expected value of the portfolio, if $V^\lambda(t,x)$ were higher than the current portfolio value of X_t, then the portfolio value would increase if the investor waits. In other words, $V^\lambda(t,x) = x$ implies the time to attain the expected maximum value of X. The boundary $b^x(t)$ given in (4) identifies the maximum value since $V^\lambda(t,b^x(t)) = b^x(t)$.

Eq. (3) yields

$$e^{-\lambda V^\lambda(t,x)} = e^{-\lambda x} + \mathbb{E}_t\left[\int_0^{T-t} de^{-\lambda e^{-\rho u}X_u^{t,x}} I(X_u^{t,x} < b^x(t+u))\right].$$

The above implies a structure similar to the certainty equivalent by regarding the integrand of the right-hand side as the accumulated expected utility of the exponential utility with risk aversion λ and the value function $V^\lambda(t,x)$ as the certainty equivalent.

4. Numerical examples

In this section, we present a few numerical examples to illustrate the properties of our model. Here, we assume that we only know that the OU process gives the form of the portfolio value process and that we might mistakenly estimate the model parameters. A larger λ value implies anxiety regarding the agent's misspecification of the parameters. The minimum value $\lambda = 0.0$ means that the agent is perfectly confident in his or her estimation.

Let us consider the case in which the actual portfolio value process is driven by $dX_t = -40X_t dt + 0.15dB_t$. The other parameters are $\rho = 0.01$ and $T = 1$. First, we assume that we can precisely

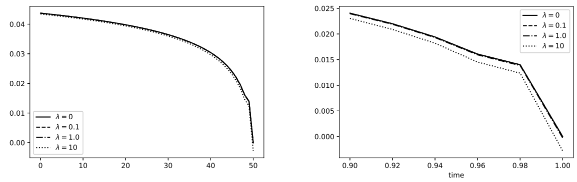

Figure 1: Optimal boundaries when agents know the true parameters.

estimate the parameters, i.e., that our estimates are $\hat{\alpha} = 40, \hat{\mu} = 0.0$ and $\hat{\sigma} = 0.15$.

Next, we consider the case in which the agent mistakenly estimates model parameter α, which is the speed of mean reversion. Figure 2 shows two cases for this.

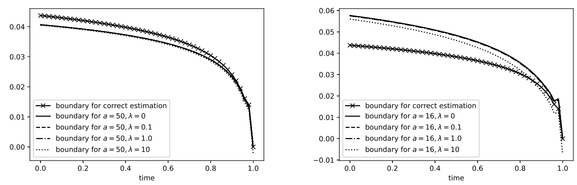

Figure 2: Optimal boundaries when agents mistakenly estimate parameter α, where the "boundary for correct estimation" is the optimal boundary with $\lambda = 0$ for the true parameters. The left panel shows boundaries using the estimated $\hat{\alpha} = 50$, and the right panel shows those using the estimated $\hat{\alpha} = 16$ for the true value $\alpha = 40$.

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