Continuous–time optimal execution in a Markovian environment

非会員　大阪大学 大学院基礎工学研究科　深澤 正彰　FUKASAWA Masaaki
大阪大学 大学院数理・データ科学教育研究センター
01007984　大和大学 政治経済学部　大西 匡光　OHNISHI Masamitsu
大阪大学 大学院数理・データ科学教育研究センター
05001073　東京都立大学 大学院経営学研究科　*下清水 慎　SHIMOSHIMIZU Makoto

1. Introduction

This study examines a continuous–time optimal execution problem incorporating an exogenous effect called a *Markovian environment*. We derive an optimal execution strategy and show that the Markovian environment described by an Ornstein-Uhlenbeck (OU) process directly affects the execution strategy.

2. Market Model

We assume a (risk–averse) large trader with the risk–aversion parameter $\gamma > 0$ in a financial market. He/she is required to purchase $\mathfrak{Q}(\in \mathbb{R})$ volume of a risky asset in the time window $[0, T]$. Let $Q_t(\in \mathbb{R})$ be the cumulative purchase up to time $t \in [0, T]$ of the large trader. Then, the number of shares remained to purchase at time $t \in [0, T]$ is described as

$$\overline{Q}_t = \mathfrak{Q} - Q_t, \tag{1}$$

with the initial and terminal conditions: $\overline{Q}_0 = \mathfrak{Q}$ and $\overline{Q}_T = 0$. We consider a continuous trading strategy:

$$\mathrm{d}Q_t = \dot{Q}_t \mathrm{d}t. \tag{2}$$

Here it is assumed that Q_t is continuously differentiable in time $t \in [0, T]$. The execution price of an asset $\widehat{P}$ is assumed to follow a *linear price impact model*

$$\widehat{P}_t = P_t + \lambda_t \dot{Q}_t, \tag{3}$$

where $P_t(\in \mathbb{R})$ represents the market price of the asset and λ_t the price impact coefficient at time $t \in [0, T]$.

The wealth process, denoted by W_t, evolves as

$$\mathrm{d}W_t = -\widehat{P}_t \mathrm{d}Q_t = -\left(P_t + \lambda_t \dot{Q}_t\right) \dot{Q}_t \mathrm{d}t. \tag{4}$$

We define the following market price dynamics:

$$\mathrm{d}P_t = \beta_t \lambda_t \dot{Q}_t \mathrm{d}t + \mathcal{I}_t \mathrm{d}t + \mathrm{d}Z_t. \tag{5}$$

$\beta_t \lambda_t \dot{Q}_t$ represents the *permanent impact*. Z_t stands for the effect of *public information* about the economic situation. We additionally assume that a *Markovian environment*, described by $\mathcal{I}_t$, affects the asset price.

Remark 1. The Markovian environment can characterize aggregate orders submitted by *small traders*, the effect of *order book imbalance* or *order flow imbalance*, the *difference of the returns* between two assets, and the effect of *trading activity of another asset*, etc.

The mathematically formal setting is as follows. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space, where the processes of $\mathcal{I}_t$ and Z_t are defined as follows:

$$\mathrm{d}\mathcal{I}_t = \left(a_t^{\mathcal{I}} - b_t^{\mathcal{I}} \mathcal{I}_t\right) \mathrm{d}t + \sigma_t^{\mathcal{I}} \mathrm{d}B_t^{\mathcal{I}}; \tag{6}$$

$$\mathrm{d}Z_t = \mu_t^Z \mathrm{d}t + \sigma_t^Z \mathrm{d}B_t^Z, \tag{7}$$

where $B_t^{\mathcal{I}}$ and B_t^Z stand for standard Brownian motions with $B_0^{\mathcal{I}} = 0$, $B_0^Z = 0$. The filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is generated by (B_t^v, B_t^Z) and satisfies the usual conditions: $\mathcal{F}_t = \sigma\left\{(B_s^{\mathcal{I}}, B_s^Z), s \in [0, t]\right\}$. We assume that the quadratic co–variation of $B_t^{\mathcal{I}}$ and B_t^Z takes the following form with correlation coefficient $\rho^{Z,\mathcal{I}}$:

$$\mathrm{d}\langle B^{\mathcal{I}}, B^Z \rangle_t = \rho^{\mathcal{I},Z} \mathrm{d}t, \tag{8}$$

Here $a_t^{\mathcal{I}}, b_t^{\mathcal{I}}, \mu_t^Z, \sigma_t^{\mathcal{I}}, \sigma_t^Z$ are all deterministic in time t.

If we assume that the information flow accessible for the large trader is carried by the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, the executed volume Q_t of the large trader by time $t \in [0, T]$ is an $\mathcal{F}_t$–measurable (real–valued) random variable. Let $\mathcal{A}$ be the set of admissible execution strategies. The dynamics of each variable depend on the process of the cumulative purchase (or liquidation) denoted by $Q = \{Q_s\}_{s \in [0,t]}$:

$$\begin{aligned}
\mathrm{d}W_t^Q &= -\widehat{P}_t^Q \mathrm{d}Q_t = -\left(P_t^Q + \lambda_t \dot{Q}_t\right) \dot{Q}_t \mathrm{d}t; \\
\mathrm{d}P_t^Q &= \beta_t \lambda_t \dot{Q}_t \mathrm{d}t + \mathcal{I}_t \mathrm{d}t + \mathrm{d}Z_t; \\
\mathrm{d}\overline{Q}_t^Q &= -\mathrm{d}Q_t = -\dot{Q}_t \mathrm{d}t;
\end{aligned}$$

To simplify the notations, we suppress the superscript Q and simply use the ones defined in the previous description except for the cases when we should emphasize the time–dependence explicitly.

3. Performance Criteria

3.1. A Hard Constraint

The state, denoted by $\boldsymbol{s}_t$, is defined as 4–tuple:

$$\boldsymbol{s}_t := \left(W_t, P_t, \overline{Q}_t, \mathcal{I}_t\right)^\top \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} =: S. \quad (9)$$

The utility function of the large trader is assumed to take the form of a Constant Absolute Risk Aversion (CARA) von Neumann-Morgenstern (vN-M) utility function. The utility payoff (or reward) arises only from the terminal wealth at maturity:

$$g_T(\boldsymbol{s}_T) := \begin{cases} -\exp\{-\gamma W_T\}, & \text{if } \overline{Q}_T = 0; \\ -\infty, & \text{if } \overline{Q}_T \neq 0. \end{cases} \quad (10)$$

We define the (conditional) expected utility at time $t \in [0,T]$ on an execution strategy $Q = \{Q_t\}_{0\le t\le T}$ as

$$V_t^Q := \mathbb{E}\left[g_T(\boldsymbol{s}_T^Q)\middle|\mathcal{F}_t\right] \quad t\in[0,T]. \quad (11)$$

Let the optimal value from time $t \in [0,T]$ by

$$V_t := \operatorname*{ess\,sup}_{Q\in\mathcal{A}} V_t^Q, \quad t\in[0,T]. \quad (12)$$

Then V_t depends on the history or information $\mathcal{F}_t$ only through the (controlled) state: $\boldsymbol{s}_t \in S$. We denote this functional dependence as follows:

$$V\left[t, W_t, P_t, \overline{Q}_t, \mathcal{I}_t\right] := V_t, \quad t\in[0,T]. \quad (13)$$

3.2. A Soft Constraint

We consider a model that a large trader can execute his/her remaining execution volume at the terminal with closing price plus an additive cost χ_T per unit.

The value function at maturity then becomes

$$V\left[T,\boldsymbol{s}_T\right] = -\exp\Big\{-\gamma\big[W_T - (P_T + \chi_T\overline{Q}_T)\overline{Q}_T\big]\Big\}, \quad (14)$$

and the conditional expected utility at time $t \in [0,T]$

$$V_t^Q := \mathbb{E}\left[-\exp\Big\{-\gamma\big[W_T - (P_T + \chi_T\overline{Q}_T)\overline{Q}_T\big]\Big\}\middle|\mathcal{F}_t\right]. \quad (15)$$

3.3. HJB Equation

The optimal value function $V_t\left[\boldsymbol{s}_t\right]$ with the terminal condition (14) satisfies, from the dynamic programming principle, the Hamilton–Jacobi–Bellman (HJB) equation for the optimal execution speed $\dot{Q}$:

$$\sup_{\dot{Q}_t\in\mathbb{R}}\Big[-\Big(P_t+\lambda_t\dot{Q}_t\Big)\dot{Q}_t\partial_W V + \beta_t\lambda_t\dot{Q}_t\partial_P V - \dot{Q}_t\partial_{\overline{Q}}V\Big]$$
$$+\partial_t V + \left(\mathcal{I}_t + \mu_t^Z\right)\partial_P V + (a_t^{\mathcal{I}} - b_t^{\mathcal{I}}\mathcal{I}_t)\partial_{\mathcal{I}}V + \frac{1}{2}\Big\{(\sigma_t^Z)^2\partial_{PP}V$$
$$+2\sigma_t^{\mathcal{I}}\sigma_t^Z\rho^{\mathcal{I},Z}\partial_{P\mathcal{I}}V + (\sigma_t^{\mathcal{I}})^2\partial_{\mathcal{I}\mathcal{I}}V\Big\} = 0, \quad (16)$$

if we assume the function $V\colon [0,T]\times S \to \mathbb{R}$ is in $\mathcal{C}^{1,2}$.

4. Optimal Execution Strategy

Theorem 1. Under a set of regularity conditions,

1. the optimal execution speed at time $t \in [0,T]$, denoted as $\dot{Q}_t^*$, becomes an *affine function* of $\overline{Q}_t$ and $\mathcal{I}_t$, and *independnet of* W_t *and* P_t:

$$\dot{Q}_t^* = a_t + b_t\overline{Q}_t + c_t\mathcal{I}_t, \quad (17)$$

   where a_t, b_t, c_t are deterministic functions of time;

2. the optimal value function $V[t,\boldsymbol{s}_t]$ at time $t \in [0,T]$ is represented as follows:

$$V[t,\boldsymbol{s}_t] = -\exp\Big\{-\gamma\Big[W_t - P_t\overline{Q}_t + G_t\overline{Q}_t^2 + H_t\overline{Q}_t + M_t\overline{Q}_t\mathcal{I}_t + X_t\mathcal{I}_t^2 + Y_t\mathcal{I}_t + K_t\Big]\Big\}, \quad (18)$$

   where $G_t, H_t, M_t, X_t, Y_t, K_t$ are deterministic functions of time, and these are assumed to exist as a unique solution of a simultaneous system of ordinary differential equations.

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