Exact Solution Methods for the p-Median Problem with Manhattan Distance and a River with Crossings

University of Strathclyde *Thomas Byrne
01204223 Nanzan University Atsuo Suzuki

1. Introduction

Rivers are a source of fresh, drinkable water, carrying and distributing important salts and nutrients. In supporting plant and animal life, rivers bestow a useful food source as well allow the use of irrigation, essential to food production. Vegetation that thrives around rivers facilitates lower air and surface temperatures by releasing moisture into the atmosphere and providing shade. Additionally, rivers are often crucial for transportation and commerce and more recently have been developed to produce electricity and to provide for leisure activities.

Not surprisingly, large population densities congregate around large rivers. While humans inhabit around 38% of the world's surface area, humans inhabit over 40% of the area for which a river is the closest water feature. In fact, on average, the closest body of water to a human is a large river at a median distance of 2.2km [1].

However, in current facility location models, the pivotal influence on travel that a river exerts has not yet been adequately addressed. Despite providing such vital resources to a settlement to this day, rivers are usually impassable but for preconstructed crossings and so present a very real barrier to anyone hoping to travel besides or to the other side of the river.

We propose an exact solution method for the p-median problem with Manhattan (l_1) distance and a river, a problem we shall refer to as pMPL1(R,C). pMPL1(R,C) requires the locating of several facilities in order to minimise the total travelling distance from the given demand points to their nearest facility. For any route across the river, a crossing point is chosen from a given set of crossings so as to minimise the shortest paths where distance is measured using the Manhattan metric which, taking into account the grid structure prevalent in many cities' transport infrastructure, is most representative when considering urban applications of facility location.

The objective function of pMPL1(R,C) is not convex and there are many local minima. Resorting to the existing heuristic methods risks obtaining not the exact solution but one of these local minima. A naïve solution method is to enumerate all the candidate points and to evaluate the objective function at each in order to find the solution. This, not surprisingly, is time-consuming. Instead, we construct a BTST algorithm [2] for pMPL1(R,C) which obtains the exact solution in practical computational time. We compare our algorithm with the naïve enumeration method and show the effectiveness of our algorithm.

2. The p-median problem with Manhattan distance and a river

For our setup, consider a river represented by a monotone line R across which travel is prohibited but upon which there are k crossings $c_i \in R$ ($C = \{c_1, ..., c_k\}$) through which travel is permitted. Due to the monotonicity of R, the lengths of shortest paths between points on the same side of R are unaffected and the shortest path between points either side of R will pass through exactly one crossing. Therefore, the distance between $Z_1 = (x_1, y_1)$ and $Z_2 = (x_2, y_2)$ on the same side of R is $d_1^{R,C}(Z_1, Z_2) = d_1(Z_1, Z_2) = |x_1 - x_2| + |y_1 - y_2|$ and $d_1^{R,C}(Z_1, Z_2) = \min_i\{d_1(Z_1, c_i) + d_1(c_i, Z_2)\}$ if they are on opposite sides of R.

Let us now formally state the p-median problem with Manhattan distance, a river R, and crossings C. We have a set of $n \in \mathbb{N}$ demand points with locations $P_i = (a_i, b_i) \in \mathbb{R}^2$ and with weights $w_i (> 0)$, $i = 1, \ldots, n$. Each customer is assumed to obtain service from the closest facility, measured using the l_1-metric across R at C described above. The goal is to obtain the exact p locations $Z_j = (x_j, y_j)$, $j = 1, ..., p$, within some convex, polygonal feasible region $FR \subset \mathbb{R}^2$, which minimise the objective function

$$\sum_{i=1}^{n} w_i \min_j \{d_1^{R,C}(P_i, Z_j)\} \,. \qquad (1)$$

This is the weighted sum of the distances between customers and their closest facility.

While this work concerns the p-median problem, it turns out that much can be learnt from the 1-median problem. For any optimal solution to the p-median problem, each facility must also be the optimal solution to the 1-median problem presented by the demand points for which it is the closest facility. Thus we explore properties of solutions to the 1-median problem as set up previously but with the objective of finding the location Z which minimises

$$\sum_{i=1}^{n} w_i d_1^{R,C}(P_i, X), \tag{2}$$

before combining this theory to present an exact algorithm for the solution of pMPL1(R,C).

To this end, let $X = \{a_1, \ldots, a_n, c_{1x}, \ldots, c_{kx}\}$ and $Y = \{b_1, \ldots, b_n, c_{1y}, \ldots, c_{ky}\}$ and sort these sets as $x_{(1)} \leq \ldots \leq x_{(n+k)}$ and $y_{(1)} \leq \cdots \leq y_{(n+k)}$ respectively. The optimal solution lies within the rectangle $[x_{(1)}, x_{(n+k)}] \times [y_{(1)}, y_{(n+k)}]$ so we need only consider the intersection of this rectangle with FR as our feasible region. We consider $(n+k-1)^2$ 'tiles' $[x_{(i)}, x_{(i+1)}] \times [y_{(j)}, y_{(j+1)}], i, j = 1, ..., n+k-1$, and call the points $[x_{(i)}, y_{(j)}], i, j = 1, ..., n+k-1$, grid points.

It can be proven that the objective function (2) is a linear function within each tile. From this, we obtain the following result.

Theorem 1: *For any polygonal shape T, there exists an optimal location for a facility within T as part of an exact solution to pMPL1(R,C) at either a vertex of T, a grid point within T, or an intersection point of an edge of T and a grid line.*

3. BTST algorithms for pMPL1(R,C)

The BTST-style algorithm for pMPL1(R,C) is described as follows for a general chosen shape:

1) Tessellate the feasible region into shapes of the chosen form.
2) Make a list of the sets of p shapes.
3) Calculate an upper bound UB for (1).
4) For each set S of p shapes in the list, calculate a lower bound LB^S for (1) where each facility is confined to each shape in S.
5) If $LB^S > UB/(1+\epsilon)$, remove set S from the list.
6) Branch and bound
   - A) Choose the set with the lowest LB^S. Divide each constituent shape into similar shapes. Form new sets of p shapes from these and calculate LB^S for each new set, updating UB if possible.
   - B) If $LB^S > UB/(1+\epsilon)$, remove set S from the list.
   - C) If the list is empty, UB is the optimal value and the optimal solution lies within the associated set of p shapes (gravity centres of the shapes can be used). Otherwise, return to A.

We calculate UB by evaluating (1) at the gravity centres of the p candidate triangles. To calculate LB for a set of p triangles T_1, ..., T_p, we transform (1) into $p+1$ terms as follows:

$$\sum_{i=1}^{n} w_i \min_{k=1,\ldots,p} d_1^{R,C}(X_k, P_i) \tag{3}$$

$$= \sum_{k=1}^{p} \sum_{i \in I_k} w_i d_1^{R,C}(X_k, P_i)$$

$$+ \sum_{i \in I_{p+1}} w_i \min_{k=1,\ldots,p} w_i d_1^{R,C}(X_k, P_i) \tag{4}$$

where $I = \{1, ..., n\}$, $I_k = \{i \in I | d_1^{R,C}(t_k, P_i) < d_1^{R,C}(t_{k'}, P_i), \forall t_k \in T_k, t_{k'} \in T_{k'}, k' (\neq k) = 1, ..., p\}$ $(k = 1, ..., p)$, and $I_{p+1} = I \backslash (\cup_{k=1}^{p} I_k)$.

Theorem 1 gives the exact solution X_k within T_k for the first term of (4) for each k. The second term is evaluated by the following equation:

$$\sum_{i \in I_{p+1}} w_i \min_{k=1,\ldots,p} w_i d_1^{R,C}(X_k, P_i) \tag{5}$$

$$\geqq \sum_{i \in I_{p+1}} w_i \min_{k=1,\ldots,p} w_i d_1^{R,C}(T_k, P_i) \tag{6}$$

where $d_1^{R,C}(T_k, P_i) = \min_{t_k \in T_k} d_1^{R,C}(t_k, P_i)$.

References

[1] Kummu, M., de Moel, H., Ward, P. J., & Varis, O. (2011). How close do we live to water? A global analysis of population distance to freshwater bodies. *PloS one*, 6(6), e20578.

[2] Drezner, Z., & Suzuki, A. (2004). The big triangle small triangle method for the solution of non-convex facility location problems. *Operations Research*, 52: 128–135.