Maximum Russell graph measures and extended production possibility sets

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1. Introduction

Data envelopment analysis (DEA) is a mathematical programming method to measure the relative efficiency of production activities. Recently, the concept of similarity of the projection point has been incorporated into the studies of least-distance DEA. The extended facet production possibility set (EFPPS) is often used in the least-distance DEA to ensure monotonicity. This study shows that the EFPPS allows *free lunch*. We also propose a maximum Russell graph measure DEA model (i) satisfies strong monotonicity; (ii) provides the closest target; (iii) can be solved using linear programming (LP).

2. Extended facet production possibility set

Aparicio and Pastor [1] defined a new version of the traditional output-oriented Russell graph measure for determining the least distance from the assessed DMU to the strongly efficient frontier $\partial^s(P)$ of a production possibility set $P \subseteq \mathbb{R}_+^{m+s}$, where $\partial^s(P) :=$

$$\left\{(\boldsymbol{x},\boldsymbol{y})\in P \;\middle|\; \begin{array}{l}(-\boldsymbol{x},\boldsymbol{y})\leq(-\boldsymbol{x}',\boldsymbol{y}')\\ (-\boldsymbol{x},\boldsymbol{y})\neq(-\boldsymbol{x}',\boldsymbol{y}')\end{array} \implies (\boldsymbol{x}',\boldsymbol{y}')\notin P\right\}.$$

By relaxing the above definition, the weakly efficient frontier of P is given by $\partial^w(P) :=$

$$\left\{(\boldsymbol{x},\boldsymbol{y})\in P \;\middle|\; (-\boldsymbol{x},\boldsymbol{y})<(-\boldsymbol{x}',\boldsymbol{y}') \implies (\boldsymbol{x}',\boldsymbol{y}')\notin P\right\}.$$

They also proved that it is possible to ensure strong monotonicity by using the extension of the full-dimensional efficient facets (FDEFs).

Given a polyhedron $P \subseteq \mathbb{R}_+^{m+s}$, a face of F of P is called a FDEF if $F \subseteq \partial^s(P)$ and $\dim F = m+s-1$. Suppose that P has an FDEF and let $F_1, \cdots, F_K$ be all the FDEFs of P. For each $k = 1, \cdots, K$, since F_k is an FDEF of P, there exist $v_i^k > 0$ $(i = 1, \ldots, m)$, $u_r^k > 0$ $(r = 1, \ldots, s)$, and $\psi^k \in \mathbb{R}$ such that

$$F_k = P \cap \left\{(\boldsymbol{x},\boldsymbol{y}) \;\middle|\; -\sum_{i=1}^m v_i^k x_i + \sum_{r=1}^s u_r^k y_r = \psi^k\right\},$$
$$P \subseteq \left\{(\boldsymbol{x},\boldsymbol{y}) \;\middle|\; -\sum_{i=1}^m v_i^k x_i + \sum_{r=1}^s u_r^k y_r \leq \psi^k\right\}.$$

We define

$$P_{\scriptscriptstyle EXFA} = \left\{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}_+^{m+s} \;\middle|\; \begin{array}{l} -\sum_{i=1}^m v_i^k x_i + \sum_{r=1}^s u_r^k y_r \leq \psi^k \\ (k = 1, \ldots, K)\end{array}\right\}.$$

If there exists $\boldsymbol{y} \in \mathbb{R}_+^s \setminus \{0\}$ such that $(\boldsymbol{0},\boldsymbol{y}) \in P$, then the production possibility set P allows free lunch and we call $(\boldsymbol{0},\boldsymbol{y})$ a free-lunch vector. Note that $P_{\scriptscriptstyle EXFA}$ may allow free lunch.

3. Issues of free lunch

The following counterexample shows that the EFPPS may allow free lunch. We proceed with the discussion under the assumption of various returns-to-scale.

Table 1: A numeric example

	A	B	C	D	E
x	1	2	5	3	4
y	4	5	6	2	2

The efficient DMUs consist of DMU_A, DMU_B, and DMU_C. The strongly frontier of P consists of two line segments, that is

$$\partial^s(P) = \{(x,y) \mid x - y + 3 = 0, 1 \leq x \leq 2\}$$
$$\cup \{(x,y) \mid x - 3y + 13 = 0, 2 \leq x \leq 5\},$$

and

$$P = \left\{(x,y) \;\middle|\; \begin{array}{l} \sum_{j=A}^C \lambda_j (x_j, -y_j) \leq (x, -y) \\ \sum_{j=A}^C \lambda_j = 1, y \geq 0 \\ \lambda_j \geq 0 \ (j = A, B, C)\end{array}\right\}.$$

We have $(0,0) \notin P$ and hence, $(0,y) \notin P$ for any $y > 0$. On the other hand, the EFPPS can be represented as

$$P_{\text{EXFA}} = P_{\text{CON}} \cap \mathbb{R}^2_+,$$

where

$$P_{\text{CON}} = \left\{ (x,y) \;\middle|\; \begin{array}{l} x - y + 3 \geq 0, \\ x - 3y + 13 \geq 0 \end{array} \right\}$$

is convex. The strongly and weakly efficient frontiers of P_{EXFA} are given by

$$\partial^s(P_{\text{EXFA}}) = \partial^s(P_{\text{CON}}) \cap \mathbb{R}^2_+,$$
$$\partial^w(P_{\text{EXFA}}) = \partial^s(P_{\text{EXFA}}) \cup \{(0,y) \mid 0 \leq y \leq 3\},$$

respectively. By the free-lunch vector $(0,y) \in P_{\text{EXFA}}$ for any positive $y \leq 3$, P_{EXFA} allows the free lunch.

4. Maximum Russell graph measure DEA model

The Rusell graph measure DEA model [2] is

$$\min \quad \frac{1}{m+s}\left(\sum_{i=1}^{m}(1-\delta_i^-) + \sum_{r=1}^{s}\frac{1}{1+\delta_r^+}\right) \quad (1)$$
$$\text{s.t.} \quad (\boldsymbol{x} - N(\boldsymbol{x})\boldsymbol{\delta}^-, \boldsymbol{y} + N(\boldsymbol{y})\boldsymbol{\delta}^+) \in P_{\text{EXFA}}, \quad (2)$$
$$\boldsymbol{0} \leq \boldsymbol{\delta}^- \leq \boldsymbol{e}, \; \boldsymbol{0} \leq \boldsymbol{\delta}^+ \quad (3)$$

where $N(\boldsymbol{z})$ is a diagonal matrix whose (p,p) entry is z_p and $\boldsymbol{e}$ is all one vector. Let $f(\boldsymbol{x}, \boldsymbol{y})$ be the optimal value of the DEA model (1)–(3). Then, f is a Russell graph measure on $P_{\text{EXFA}} \cap \mathbb{R}^{m+s}_{++}$. Using the example of section 3, we have

$$f(3,2) = f(4,2) = \frac{1}{3},$$

which means that the Rusell graph measure f does not satisfy the strong monotonicity.

A maximum Rusell graph measure DEA model is defined as

$$\max \quad \frac{1}{m+s}\left(\sum_{i=1}^{m}(1-\delta_i^-) + \sum_{r=1}^{s}\frac{1}{1+\delta_r^+}\right) \quad (4)$$
$$\text{s.t.} \; (\boldsymbol{x} - N(\boldsymbol{x})\boldsymbol{\delta}^-, \boldsymbol{y} + N(\boldsymbol{y})\boldsymbol{\delta}^+) \in \partial^s(P_{\text{EXFA}}) \quad (5)$$
$$\boldsymbol{0} \leq \boldsymbol{\delta}^- \leq \boldsymbol{e}, \; \boldsymbol{0} \leq \boldsymbol{\delta}^+ \quad (6)$$

for any $P_{\text{EXFA}} \cap \mathbb{R}^{m+s}_{++}$. Let $\mathscr{F}$ be the optimal value of the DEA model (4)–(6). Then, $\mathscr{F}$ is called a maximum Rusell graph measure and the corresponding DEA model has the following properties:

Theorem 4.1. *The efficiency measure $\mathscr{F}$ is strongly monotonic on $P_{\text{EXFA}} \cap \mathbb{R}^{m+s}_{++}$. Suppose any $(\boldsymbol{x}, \boldsymbol{y}) \in P_{\text{EXFA}} \cap \mathbb{R}^{m+s}_{++}$ and let $(\boldsymbol{\delta}^{-*}, \boldsymbol{\delta}^{+*})$ be an optimal solution of (4)–(6). Then, $(\boldsymbol{x} - N(\boldsymbol{x})\boldsymbol{\delta}^{-*}, \boldsymbol{y} + N(\boldsymbol{y})\boldsymbol{\delta}^{+*})$ is not a free-lunch vector.*

The commensurable Hölder input distance function for the measurement of inefficiency is defined as

$$\min \quad \sum_{i=1}^{m} \delta_i^- \quad (7)$$
$$\text{s.t} \quad (\boldsymbol{x} - N(\boldsymbol{x})\boldsymbol{\delta}^-, \boldsymbol{y}) \in \partial^w(P_{\text{EXFA}}), \quad (8)$$
$$\boldsymbol{0} \leq \boldsymbol{\delta}^-, \quad (9)$$

and the commensurable Hölder output distance function for the measurement of inefficiency is defined as

$$\min \quad \sum_{r=1}^{s} \delta_r^+ \quad (10)$$
$$\text{s.t} \quad (\boldsymbol{x}, \boldsymbol{y} + N(\boldsymbol{y})\boldsymbol{\delta}^+) \in \partial^w(P_{\text{EXFA}}), \quad (11)$$
$$\boldsymbol{0} \leq \boldsymbol{\delta}^+. \quad (12)$$

Let $D^-(\boldsymbol{x}, \boldsymbol{y})$ be the commensurable Hölder input distance function for $(\boldsymbol{x}, \boldsymbol{y})$. Then, D^- can be obtained by solving the m LP problems: $\max\{\delta \mid (\boldsymbol{x} - x_i\delta\boldsymbol{e}_i, \boldsymbol{y}) \in P_{\text{EXFA}}\}$ $(i = 1, \ldots, m)$. Let $D^+(\boldsymbol{x}, \boldsymbol{y})$ be the commensurable Hölder output distance function for $(\boldsymbol{x}, \boldsymbol{y})$. Then, D^+ can be obtained by solving the s LP problems. Ths maximum Rusell graph measure satisfies $\mathscr{F}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{m+s} \times$

$$\max\left\{ m + s - D^-(\boldsymbol{x}, \boldsymbol{y}), \; m + s - \frac{D^+(\boldsymbol{x}, \boldsymbol{y})}{1 + D^+(\boldsymbol{x}, \boldsymbol{y})} \right\},$$

which means that the maximum Rusell graph measure can be solved by LPs.

References

[1] Aparicio, J., & Pastor, J. T. (2014) Closest targets and strong monotonicity on the strongly efficient frontier in DEA. Omega, 44, 51-57.

[2] Färe, R., Grosskopf, S., & Lovell, C. A. (1985). Nonradial Efficiency Measures. In The Measurement of Efficiency of Production (pp. 141-162). Springer, Dordrecht.