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# LP approach to the least-distance efficiency of nonlinear DEA models

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## 1. Introduction

Data envelopment analysis (DEA) is a mathematical programming approach for evaluating relative efficiency for decision-making units (DMUs). The DEA approach can also provide the benchmarking information for each DMU by deriving its projection point. Recently, the concept of similarity of the projection point has been incorporated into the studies on the least-distance DEA. The least-distance DEA models aims to find the closest target for the evaluated DMU. Methodologically, several least-distance DEA models have been proposed that reverse the optimization of the conventional DEA model.

Denote by “maximum efficiency measure” the optimal value solved by such models. If there exists a maximum efficiency measure that satisfies the units invariance and monotonicity and its corresponding projection point coincides with the closest target for all DMUs, then the maximum efficiency measure is suitable from both the practical and axiomatic viewpoints. However, the axiomatic approach to least-distance inefficiency measures shows the impossibility of monotonicity (see Ando et al. (2012) for the details). To ensure monotonicity, Aparicio & Pastor (2014) incorporated an extended efficient facet approach into the least-distance DEA. Because the extended efficient facet approach relies on the mixed-integer linear programming (MILP), it is unsuitable for the DEA models that have nonlinear objective functions. To solve this issue, This study proposes a class of nonlinear DEA models, including variants of the Russell graph measure (RM), BRWZ measure, slack-based measure (SBM), and geometric distance function (GDF).

## 2. Issue of monotonicity

Because the nonlinearity of their objective functions appears only on an output side, we examine all the maximum efficiency measures of the class in the output-orientation. The output-oriented objective function of RM is

$$g^R(\boldsymbol{\delta}) \;:=\; \frac{1}{s}\left(\sum_{r=1}^{s}\frac{1}{1+\delta_r}\right), \tag{1}$$

which is equivalent to the output-oriented objective function of the BRWZ measure. The output-oriented objective function of SBM is

$$g^S(\boldsymbol{\delta}) \;:=\; \frac{1}{1+\frac{1}{s}\sum_{r=1}^{s}\delta_r}. \tag{2}$$

The output-oriented objective function of GDF is

$$g^G(\boldsymbol{\delta}) \;:=\; \left(\prod_{r=1}^{s}\frac{1}{1+\delta_r}\right)^{\frac{1}{s}}. \tag{3}$$

The following counterexample shows that the output-oriented maximum RM, BRWZ measure, SBM, and GDF do not satisfy the weak monotonicity.

表 1: A counterexample

| DMU | $x_1$ | $y_1$ | $y_2$ | $y_3$ | status |
|---|---|---|---|---|---|
| $A$ | 1 | 10 | 6 | 80 | Eff. |
| $B$ | 1 | 8 | 12 | 12 | Eff. |
| $C$ | 1 | 8 | 6 | 8 | Ineff. |
| $D$ | 1 | 10 | 6 | 8 | Ineff. |

## 3. Generalized output-oriented maximum efficiency measure

Theorem 3.1 shows the common properties among the functions (1), (2), and (3):

**Theorem 3.1.** *All the $g^R$, $g^S$, and $g^G$ are decreasing, continuous, and quasiconvex on $\mathbb{R}^s_+$.*

Let $g$ be a decreasing, continuous, and quasiconvex function on $\mathbb{R}^s_+$. Consider the following output-oriented DEA model:

$$\begin{aligned} \max \quad & g(\boldsymbol{\delta}) && (4)\\ \text{s.t.} \quad & (\boldsymbol{x}, \boldsymbol{y} + N(\boldsymbol{y})\boldsymbol{\delta}) \in \partial^w(P) && (5)\\ & \boldsymbol{\delta} \geq \boldsymbol{0}, && (6) \end{aligned}$$

where $\partial^w(P)$ is the weakly efficient frontier and $P$ represents the conventional production possibility set. Let $f(\boldsymbol{x}, \boldsymbol{y})$ be the optimal value of model (4)–(6) for $(\boldsymbol{x}, \boldsymbol{y}) \in P \cap \left(\left(\mathbb{R}^m_+ \setminus \{\boldsymbol{0}\}\right) \times \mathbb{R}^s_{++}\right)$. Therefore, $f$ is an efficiency measure over $P \cap \left(\left(\mathbb{R}^m_+ \setminus \{\boldsymbol{0}\}\right) \times \mathbb{R}^s_{++}\right)$ if $g : \mathbb{R}^s_+ \to (0, 1]$.

Theorems 3.2 and 3.3 show that efficiency measure $f$ can be computed by solving $s$ maximization problems, and it satisfies the weak monotonicity.

**Theorem 3.2.** *For any $(\boldsymbol{x}, \boldsymbol{y}) \in P \cap \left(\left(\mathbb{R}^m_+ \setminus \{\boldsymbol{0}\}\right) \times \mathbb{R}^s_{++}\right)$ and for any $r = 1, \ldots, s$, let*

$$\delta^*_r := \max\left\{ \delta \,|\, (\boldsymbol{x}, \boldsymbol{y} + \delta y_r \boldsymbol{e}_r) \in P \right\}, \quad (7)$$

*where $\boldsymbol{e}_r$ is the $r^{th}$ unit vector of $\mathbb{R}^s$. Therefore,*

$$(\boldsymbol{x}, \boldsymbol{y} + \delta^*_r y_r \boldsymbol{e}_r) \in \partial^w(P) \quad (8)$$

*and*

$$f(\boldsymbol{x}, \boldsymbol{y}) = \max\left\{ g(\delta^*_1 \boldsymbol{e}_1), \ldots, g(\delta^*_s \boldsymbol{e}_s) \right\}. \quad (9)$$

**Theorem 3.3.** *The efficiency measure $f$ is a weakly monotonic function over $P \cap \left(\left(\mathbb{R}^m_+ \setminus \{\boldsymbol{0}\}\right) \times \mathbb{R}^s_{++}\right)$.*

## 4. Properties of the projection point

It can be proved that the objective functions $g^R$, $g^S$, and $g^G$ have the same property as the $L_1$ norm property $\|\delta \boldsymbol{e}_1\|_1 = \|\delta \boldsymbol{e}_2\|_1 = \cdots = \|\delta \boldsymbol{e}_s\|_1$ for any $\delta \in R$, where $\|\boldsymbol{z}\|_1 = \sum_{r=1}^s |z_r|$ for any $\boldsymbol{z} \in \mathbb{R}^s$. That is, for any $\delta \in R_+$, $g^R(\delta \boldsymbol{e}_1) = \cdots = g^R(\delta \boldsymbol{e}_s)$; $g^S(\delta \boldsymbol{e}_1) = \cdots = g^S(\delta \boldsymbol{e}_s)$; $g^G(\delta \boldsymbol{e}_1) = \cdots = g^G(\delta \boldsymbol{e}_s)$.

The commensurable Hölder output distance function $D(\boldsymbol{x}, \boldsymbol{y})$ is

$$\min\left\{ \sum_{r=1}^s |\delta_r| \;\middle|\; (\boldsymbol{x}, \boldsymbol{y} + N(\boldsymbol{y})\boldsymbol{\delta}) \in \partial^w(P) \right\}. \quad (10)$$

An input-output vector $(\boldsymbol{x}, \boldsymbol{y} + N(\boldsymbol{y})\boldsymbol{\delta}")$ is called the closest target from $(\boldsymbol{x}, \boldsymbol{y})$ to $\partial^w(P)$ if $\boldsymbol{\delta}"$ satisfies $(\boldsymbol{x}, \boldsymbol{y} + N(\boldsymbol{y})\boldsymbol{\delta}") \in \partial^w(P)$ and $D(\boldsymbol{x}, \boldsymbol{y}) = \sum_{r=1}^s \delta"_r$.

It can be proved that the efficiency measure $f$ is a decreasing function of the least distance $D(\boldsymbol{x}, \boldsymbol{y})$. Furthermore, for any decreasing, continuous, and convex function $g$ satisfying $g(\delta \boldsymbol{e}_1) = \cdots = g(\delta \boldsymbol{e}_s), \forall \delta \in R_+$, the computation of $D(\boldsymbol{x}, \boldsymbol{y})$ provides the (i) efficiency score and (ii) projection point which is also a closest target. The computation of $D(\boldsymbol{x}, \boldsymbol{y})$ is reduced into solving the following $s$ LP problems:

$$\begin{aligned} \max \quad & \delta && (11)\\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j \boldsymbol{x}_j + \boldsymbol{d}^- = \boldsymbol{x} && (12)\\ & \sum_{j=1}^n \lambda_j \boldsymbol{y}_j - \boldsymbol{d}^+ = \boldsymbol{y} + \delta y_r \boldsymbol{e}_r && (13)\\ & \boldsymbol{d}^- \geq \boldsymbol{0},\ \boldsymbol{d}^+ \geq \boldsymbol{0},\ \boldsymbol{\lambda} \geq \boldsymbol{0}. && (14) \end{aligned}$$

Denote the optimal value of (11)–(14) by $\delta^*_r$ for all $r = 1, \ldots, s$. From Corollary 3 of Briec (1999), we obtain $D(\boldsymbol{x}, \boldsymbol{y}) = \min\{\delta^*_1, \ldots, \delta^*_s\}$. The output-oriented maximum RM, BRWZ measure, SBM, and GDF are then computed by:

$$\begin{aligned} \text{Max. RM (Max. BRWZ)} &= \frac{1}{s}\left(s - \frac{D(\boldsymbol{x}, \boldsymbol{y})}{1 + D(\boldsymbol{x}, \boldsymbol{y})}\right),\\ \text{Max. SBM} &= \frac{1}{1 + \frac{D(\boldsymbol{x}, \boldsymbol{y})}{s}},\\ \text{Max. GDF} &= \left(\frac{1}{1 + D(\boldsymbol{x}, \boldsymbol{y})}\right)^{\frac{1}{s}}, \end{aligned}$$

respectively.

## References

[1] Aparicio, J., & Pastor, J. T. (2014). Closest targets and strong monotonicity on the strongly efficient frontier in dea. Omega, 44, 51–57.