日本オペレーションズ・リサーチ学会
2022年 秋季研究発表会

Exact Solution Methods for the Multi-Weber Problem with Manhattan Distance

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1. Introduction

We propose an exact solution method for the multi-Weber problem with Manhattan (l_1) distance, a problem we shall refer to as MWPL1. MWPL1 requires the locating of several facilities in order to minimise the total travelling distance from the given demand points to their nearest facility. Distances are measured using the Manhattan metric which, taking into account the grid structure prevalent in many cities' transport infrastructure (of which Kyoto is a prime example), is most representative when considering urban applications of facility location as well as those arising from multi-dimensional data sets with heterogeneous dimensions.

It is known that the single facility Weber problem with Manhattan distance is easily solved because its objective function is a piecewise linear function. For MWPL1, a few heuristic algorithms have already been proposed. However, effective algorithms to obtain the exact solution of the MWPL1 have not been studied[1].

The algorithms we propose are based upon the BTST (Big Triangle Small Triangle) method [2], designed to obtain the exact solution of problems whose objective function is not convex. The BTST method has been applied to many continuous location problems for which the exact solutions are considered difficult to obtain.

The objective function of the MWPL1 is not convex and there are many local minima. Resorting to the existing heuristic methods risks obtaining not the exact solution but one of these local minima. A naive solution method is to enumerate all the candidate points and to evaluate the objective function at each in order to find the solution. This, not surprisingly, is time-consuming. Instead, we construct a BTST algorithm for the MWPL1 which finds the exact solution in practical computational time. We compare the algorithm with the naive enumeration method and show the effectiveness of our algorithm.

2. Exact solution of the Weber problem with Manhattan distance within a triangle

Before we describe the MWPL1, we introduce the exact solution of the Weber problem with Manhattan distance confined to a triangle.

We use the exact solution to estimate a lower bound of the objective function in BTST. It is well known that the exact solution of the Weber problem with Manhattan distance is easily obtained since the objective function is a piecewise linear function. We can divide the objective function into two terms depending on only the x-coordinates and y-coordinates respectively and find the optimal x- and y-coordinate by minimising each term independently.

However, when confined to an area not containing the convex hull of the demand points in its entirety, this optimum may not be achievable. In this case we show that the minimum is attained at a limited number of candidate points. Evaluating the objective function for each candidate point, we pick up the minimum as the exact solution.

Now we describe the problem. We have n demand points $P_i(a_i, b_i)$ whose weights are $w_i(> 0), i = 1, ..., n$. We want to obtain the exact solution $P(x, y)$ in a convex feasible region FR which minimises the objective function

$$\sum_{i=1}^{n} w_i d_M(P_i, X), \tag{1}$$

where $d_M(P_i, X) = |x - a_i| + |y - b_i|$.

We sort the x- and y-coordinates of the demand points as $a_{(1)} < \ldots < a_{(n)}$ and $b_{(1)} < \ldots < b_{(n)}$ respectively. The optimal solution lies within the rectangle $[a_{(1)}, a_{(n)}] \times [b_{(1)}, b_{(n)}]$ so we take this as FR. We consider n^2 'tiles' $[a_{(i)}, a_{(i+1)}] \times [b_{(j)}, b_{(j+1)}], i, j = 1, ..., n-1$, and call the points $[a_{(i)}, b_{(j)}], i, j = 1, ..., n-1$, grid points.

It is obvious that the objective function (1) is a linear function within each tile. From this, we obtain the following lemma concerning the optimal point within a triangle T. We obtain the similar lemma for the case of a rectangle.

Lemma 1: *The exact solution which minimises the objective function (1) in a triangle $T \subset FR$ is a vertex of T, a grid point in T, or an intersection point of an edge of T and a tile edge.*

3. BTST algorithm for MWPL1

The objective function of MWPL1 is

$$\min \quad \sum_{i=1}^{n} w_i \min_{k=1,\dots,p} d_M(X_k, P_i) \tag{2}$$

$$s.t. \quad X_k \in FR, k = 1, \dots, p \tag{3}$$

where $X_k, k = 1, \dots, p$ are the locations of the facilities to be optimised. The BTST algorithm of MWPL1 is described as follows:

1) Triangulate the convex hull of the demand points using the Delaunay triangulation.
2) Make a list of the sets of p triangles.
3) Calculate an upper bound UB for (2).
4) Calculate a lower bound LB^s for (2) for each set s of p triangles in the list.
5) If $LB^s > UB/(1+\epsilon)$, remove set s from the list.
6) Branch and bound
   A) Choose the set with the lowest LB^s and divide each constituent triangle into four similar triangles. Using these new triangles, form new sets of p triangles and calculate the LB^s for each new set, updating UB if possible.
   B) If $LB^s > UB/(1+\epsilon)$, remove set s from the list.
   C) If the list is empty, UB is the optimal value and the gravity centres of the triangles in the associated set provide the optimum solution. Otherwise, go to A.

We calculate UB by evaluating (2) at the gravity centres of the p candidate triangles. To calculate LB for a set of p triangles T_1, ..., T_p, we transform the objective function (2) into $p+1$ terms as follows:

$$\sum_{i=1}^{n} w_i \min_{k=1,\dots,p} d_M(X_k, P_i) \tag{4}$$

$$= \sum_{k=1}^{p} \sum_{i \in I_k} w_i d_M(X_k, P_i) + \sum_{i \in I_{p+1}} w_i \min_{k=1,\dots,p} w_i d_M(X_k, P_i) \tag{5}$$

where $I = \{1, \dots, n\}$, $I_k = \{i \in I | d_M(t_k, P_i) < d_M(t_{k'}, P_i), \forall t_k \in T_k, t_{k'} \in T_{k'}, k'(\neq k) = 1, \dots, p\}$ $(k = 1, \dots, p)$, and $I_{p+1} = I \backslash (\cup_{k=1}^{p} I_k)$.

To calculate the LB, we obtain the exact solution X_k within T_k for the first term of (5) using Lemma 1 for each k. The second term is evaluated by the following equation:

$$\sum_{i \in I_{p+1}} w_i \min_{k=1,\dots,p} w_i d_M(X_k, P_i) \tag{6}$$

$$\geqq \sum_{i \in I_{p+1}} w_i \min_{k=1,\dots,p} w_i d_M(T_k, P_i) \tag{7}$$

where $d_M(T_k, P_i) = \min_{t_k \in T_k} d_M(t_k, P_i)$.

4. Preliminary comparison of BTST and the naive method

In the BTST method, we triangulate FR into around $2n$ triangles. We evaluate the function value to obtain the UB for every candidate set of p triangles (of which there are ${}_{2n}C_p$) and the BTST method is effective at finding the solution because the lower bound of the objective function is tight. On the other hand, the naive enumeration method needs to evaluate every set of p candidate grid points (${}_{n^2}C_p$ solutions). The difference in complexity is $O(n^p)$ so, especially for large p, BTST will solve the problem more effectively than the naive method.

We also introduce the BRSR (Big Rectangle and Small Rectangle) method in which we use rectangles instead of triangles as in the BTST method. We implement the BTST, BRSR, and the naive methods and compare the CPU time for various values of n and p.

References

[1] Aras, N., Orbay, M., and Altinel, I. K. (2008). Efficient heuristics for the rectilinear capacitated multi-facility Weber problem. *Journal of the Operational Research Society*, 59(1): 64–79.

[2] Drezner, Z., and Suzuki, A. (2004). The big triangle small triangle method for the solution of non-convex facility location problems. *Operations Research*, 52: 128–135.