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Some properties of convex-cone-based intuitionistic fuzzy set relations

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1. Introduction

A membership function of a classical fuzzy set assigns to each element of the universe of discourse a number from the unit interval to indicate the degree of belongingness to the set under consideration. The necessity to deal with imprecision in real world problems has been a long term research challenge that has originated different extensions of fuzzy sets. Atanassov [1] introduced the concept of an intuitionistic fuzzy set (IFS) which is characterized by two functions expressing degree of belongingness and the degree of nonbelongingness, respectively. IFS can be useful to deal with situations where the classical fuzzy tools are not so efficient.

2. Preliminaries

Let Z be a real topological vector space unless otherwise specified. Let $\mathcal{P}(Z)$ denote the set of all nonempty subsets of Z. The topological interior and topological closure of a set $A \in \mathcal{P}(Z)$ are denoted by $\operatorname{int} A$ and $\operatorname{cl} A$, respectively. A set $C \in \mathcal{P}(Z)$ is a cone if $tz \in C$ for all $z \in C$ and $t > 0$. The transitive relation $\leq_C$ is induced by a convex cone C as follows: for $z, z' \in Z$, $z \leq_C z'$ if $z' - z \in C$. The eight types of set relations are defined by

$$
\begin{aligned}
A \leq_C^{(1)} B &\overset{\text{def}}{\iff} \forall a \in A, \forall b \in B, a \leq_C b;\\
A \leq_C^{(2L)} B &\overset{\text{def}}{\iff} \exists a \in A \text{ s.t. } \forall b \in B, a \leq_C b;\\
A \leq_C^{(2U)} B &\overset{\text{def}}{\iff} \exists b \in B \text{ s.t. } \forall a \in A, a \leq_C b;\\
A \leq_C^{(2)} B &\overset{\text{def}}{\iff} A \leq_C^{(2L)} B \text{ and } A \leq_C^{(2U)} B;\\
A \leq_C^{(3L)} B &\overset{\text{def}}{\iff} \forall b \in B, \exists a \in A \text{ s.t. } a \leq_C b;\\
A \leq_C^{(3U)} B &\overset{\text{def}}{\iff} \forall a \in A, \exists b \in B \text{ s.t. } a \leq_C b;\\
A \leq_C^{(3)} B &\overset{\text{def}}{\iff} A \leq_C^{(3L)} B \text{ and } A \leq_C^{(3U)} B;\\
A \leq_C^{(4)} B &\overset{\text{def}}{\iff} \exists a \in A, \exists b \in B \text{ s.t. } a \leq_C b.
\end{aligned}
$$

for $A, B \in \mathcal{P}(Z)$.

A pair $\tilde{A} = (\mu_{\tilde{A}}, \nu_{\tilde{A}})$ is called an *intuitionistic fuzzy set* or *IFS* on Z, where $\mu_{\tilde{A}}, \nu_{\tilde{A}} : Z \to [0,1]$ are the membership and non-membership functions, respectively, such that $\mu_{\tilde{A}}(z) + \nu_{\tilde{A}}(z) \leq 1$ for all $z \in Z$. When $\mu_{\tilde{A}}(z) + \nu_{\tilde{A}}(z) = 1$, $\tilde{A}$ is called a *fuzzy set* in the classical sense.

Let $I := \{(\alpha, \beta) \in [0,1]^2 \mid \alpha + \beta \leq 1\}$ be the set which is used to give values for α, β in the (α, β)-*cut* of $\tilde{A}$ defined as

$$\tilde{A}_{(\alpha,\beta)} := \{z \in Z \mid \mu_{\tilde{A}}(z) \geq \alpha \ \&\ \nu_{\tilde{A}}(z) \leq \beta\}.$$

We define $\preceq$ as the partial order where $(\alpha_1, \beta_1) \preceq (\alpha_2, \beta_2)$ means $\alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$. Clearly, $\tilde{A}_{(\alpha_2,\beta_2)} \subset \tilde{A}_{(\alpha_1,\beta_1)}$ whenever $(\alpha_1, \beta_1) \preceq (\alpha_2, \beta_2)$, for any $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in I$. $\tilde{A}$ is said to be normal if $\tilde{A}_{(\alpha,\beta)} \neq \varnothing$ for all $(\alpha, \beta) \in I$. We denote by $\mathcal{F}_{\mathcal{N}}(Z)$ the set of all normal IFS on Z.

3. Intuitionistic fuzzy set relations

By considering the set relations between (α, β)-cuts of two IFS, the intuitionistic fuzzy set relations are defined as follows.

Let $C \subset Z$ be a convex cone and $\varnothing \neq \Delta \subset I$. For each $j = 1, 2L, 2U, 2, 3L, 3U, 3, 4$, the *intuitionistic fuzzy set relation (IFSR)* $\leq_C^{\Delta(j)}$ is defined by

$$\tilde{A} \leq_C^{\Delta(j)} \tilde{B} \iff \forall (\alpha, \beta) \in \Delta, \tilde{A}_{(\alpha,\beta)} \leq_C^{(j)} \tilde{B}_{(\alpha,\beta)}$$

for normal IFS $\tilde{A}$ and $\tilde{B}$.

The set Δ is a collection of values of (α, β) that are of concern in comparing intuitionistic fuzzy sets. The minimum element $\min \Delta$ and maximum element $\max \Delta$ are considered as singleton sets provided that they exist with respect to the partial order $\preceq$.

From the definition, we obtain the following implications:

$$\tilde{A} \leq_C^{\Delta(1)} \tilde{B} \Longrightarrow \tilde{A} \leq_C^{\Delta(2L)} \tilde{B} \Longrightarrow \tilde{A} \leq_C^{\Delta(3L)} \tilde{B} \Longrightarrow \tilde{A} \leq_C^{\Delta(4)} \tilde{B};$$
$$\tilde{A} \leq_C^{\Delta(1)} \tilde{B} \Longrightarrow \tilde{A} \leq_C^{\Delta(2U)} \tilde{B} \Longrightarrow \tilde{A} \leq_C^{\Delta(3U)} \tilde{B} \Longrightarrow \tilde{A} \leq_C^{\Delta(4)} \tilde{B};$$
$$\tilde{A} \leq_C^{\Delta(1)} \tilde{B} \Longrightarrow \tilde{A} \leq_C^{\Delta(2)} \tilde{B} \Longrightarrow \tilde{A} \leq_C^{\Delta(3)} \tilde{B} \Longrightarrow \tilde{A} \leq_C^{\Delta(4)} \tilde{B};$$

for any IFS $\tilde{A}, \tilde{B}$.

If $\min \Delta$ and $\max \Delta$ exist, then

(i) $\tilde{A} \leq_C^{\Delta(1)} \tilde{B} \Longleftrightarrow \tilde{A}_{\min \Delta} \leq_C^{(1)} \tilde{B}_{\min \Delta}$;

(ii) $\tilde{A} \leq_C^{\Delta(4)} \tilde{B} \Longleftrightarrow \tilde{A}_{\max \Delta} \leq_C^{(4)} \tilde{B}_{\max \Delta}$.

As evaluation measure of the difference between two IFS, the following functions called difference evaluation functions for IFS are defined.

Let $C \subset Z$ be a convex cone, $k \in \operatorname{int} C$, and $\varnothing \neq \Delta \subset I$. For each $j = 1, 2L, 2U, 2, 3L, 3U, 3, 4$, the *difference evaluation function* $D_{C,k}^{\Delta(j)} : \mathcal{F}_{\mathcal{N}}(Z) \times \mathcal{F}_{\mathcal{N}}(Z) \to \mathbb{R} \cup \{\pm\infty\}$ is defined by

$$D_{C,k}^{\Delta(j)}(\tilde{A}, \tilde{B}) \coloneqq \sup \left\{ t \in \mathbb{R} \,\middle|\, \tilde{A} + tk \leq_C^{\Delta(j)} \tilde{B} \right\},$$

for $\tilde{A}, \tilde{B} \in \mathcal{F}_{\mathcal{N}}(Z)$.

4. Correspondences between IFSR and difference evaluation functions

Let $\varnothing \neq \Delta \subset I$. An IFS $\tilde{A}$ on a real normed space Z is said to be Δ-compact if $\tilde{A}_{(\alpha,\beta)}$ is compact for all $(\alpha, \beta) \in \Delta$. A set-valued map F is said to be *Hausdorff upper (resp. lower) continuous* at x_0 if for any neighborhood V of x_0, there is a neighborhood U of x_0 such that $F(x) \subset F(x_0) + V$ (resp. $F(x_0) \subset F(x) + V$) for all $x \in U$. Under these setting, $\tilde{A}$ is said to be

(i) *stable to Δ-level decrease* if the set-valued map $\Delta \ni (\alpha, \beta) \mapsto \tilde{A}_{(\alpha,\beta)} \in \mathcal{P}(Z)$ is Hausdorff upper continuous;

(ii) *stable to Δ-level increase* if the set-valued map $\Delta \ni (\alpha, \beta) \mapsto \tilde{A}_{(\alpha,\beta)} \in \mathcal{P}(Z)$ is Hausdorff lower continuous; see [2].

Let C be a convex cone in a real normed vector space Z, $k \in \operatorname{int} C$, $\varnothing \neq \Delta \subset I$, and $\tilde{A}, \tilde{B}$ normal IFS on Z. Then with $j = 1, 2, 3, 4$, the following statements hold:

(i) $D_{C,k}^{\Delta(j)}(\tilde{A}, \tilde{B}) > 0 \Longrightarrow \tilde{A} \leq_{\operatorname{int} C}^{\Delta(j)} \tilde{B}$;

(ii) $\tilde{A} \leq_{\operatorname{cl} C}^{\Delta(j)} \tilde{B} \Longrightarrow D_{C,k}^{\Delta(j)}(\tilde{A}, \tilde{B}) \geq 0$;

(iii) $D_{C,k}^{\Delta(1)}(\tilde{A}, \tilde{B}) \geq 0 \Longrightarrow \tilde{A} \leq_{\operatorname{cl} C}^{\Delta(1)} \tilde{B}$;

(iv) $\tilde{A} \leq_{\operatorname{int} C}^{\Delta(1)} \tilde{B} \Longrightarrow D_{C,k}^{\Delta(1)}(\tilde{A}, \tilde{B}) > 0$ if $\min \Delta$ exists and $\tilde{A}$ and $\tilde{B}$ are $(\min \Delta)$-compact;

(v) $D_{C,k}^{\Delta(2)}(\tilde{A}, \tilde{B}) \geq 0 \Longrightarrow \tilde{A} \leq_{\operatorname{cl} C}^{\Delta(2)} \tilde{B}$ if $\tilde{A}$ and $\tilde{B}$ is Δ-compact;

(vi) $\tilde{A} \leq_{\operatorname{int} C}^{\Delta(2)} \tilde{B} \Longrightarrow D_{C,k}^{\Delta(2)}(\tilde{A}, \tilde{B}) > 0$ if Δ is compact, and $\tilde{A}$ and $\tilde{B}$ are both Δ-compact and stable to Δ-level increase;

(vii) $D_{C,k}^{\Delta(3)}(\tilde{A}, \tilde{B}) \geq 0 \Longrightarrow \tilde{A} \leq_{\operatorname{cl} C}^{\Delta(3)} \tilde{B}$ if $\tilde{A}$ and $\tilde{B}$ are Δ-compact;

(viii) $\tilde{A} \leq_{\operatorname{int} C}^{\Delta(3)} \tilde{B} \Longrightarrow D_{C,k}^{\Delta(3)}(\tilde{A}, \tilde{B}) > 0$ if Δ is compact, and $\tilde{A}$ and $\tilde{B}$ are both Δ-compact and stable to Δ-level increase;

(ix) $D_{C,k}^{\Delta(4)}(\tilde{A}, \tilde{B}) \geq 0 \Longrightarrow \tilde{A} \leq_{\operatorname{cl} C}^{\Delta(4)} \tilde{B}$ if $\max \Delta$ exists and $\tilde{A}$ and $\tilde{B}$ are $(\max \Delta)$-compact;

(x) $\tilde{A} \leq_{\operatorname{int} C}^{\Delta(4)} \tilde{B} \Longrightarrow D_{C,k}^{\Delta(4)}(\tilde{A}, \tilde{B}) > 0$ if $\max \Delta$ exists.

5. Conclusion

Several types of intuitionistic fuzzy set relations have been introduced based on a convex cone as new comparison criteria of intuitionistic fuzzy sets. Several results related to the correspondences between IFS and their difference evaluation functions were obtained under some assumptions of certain compactness and stability.

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