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Ex Ante User Equilibrium of Capacitated Networks

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1 Introduction

Beckmann et al. [1] show that the flow is a Wardrop equilibrium iff it solves some optimization problem, which we call here the BMW optimization. This method has also been extended and widely applied to an equilibrium model of capacitated networks. Correa et al. [2] formally show that a solution of the constrained BMW optimization problem satisfies an extension of the uncapacitated equilibrium concept. The converse, however, does not hold true. This gap has not been addressed in the literature. We show that the constrained BMW optimization problem itself characterizes the equilibrium of some game.

To this end, we follow the approach of Marcotte et al. [3], who consider a mixed strategy formulation. In addition, we consider three stages of uncertainty. In the *ex ante* stage, a user has not been assigned his/her origin-destination (OD) pair yet. Before nature randomly assigns an OD pair to each user, the user chooses the best set of mixed strategies, each of which is a plan contingent on the assignment of his/her OD pair. Then, in the interim stage, a user knows his/her OD pair and chooses the best mixed strategy, which assigns a probability to each path associated with his/her OD pair. Finally, in the *ex post* stage, each user with a specific OD pair chooses the best path from the set of the paths associated with his/her OD pair. The change in the strategy space (from pure to mixed), the mode of equilibrium (from asymmetric to symmetric), and the decision timing have no significant implications on the equilibrium analysis of networks in the absence of link capacity constraints. However, it turns out that this seemingly superficial change helps characterize the equilibrium of constrained networks. Specifically, when there are link capacities, the *ex ante* (*ex post*) user equilibrium is the strongest (weakest). There is an intuition behind this. From the *ex post* stage to the interim, and from the interim stage to the *ex ante*, as moving backward in time, a user has more strategic maneuverability to stay within the constraints. Thus, moving backward in time, the equilibrium requirements become more stringent.

2 Model and Analysis

Let A and K be the sets of arcs and OD pairs, resp. A path P is a set of arcs connecting an OD pair. Let $\mathcal{P}$ be the set of all paths and $\mathcal{P}_k$ the set of paths associated with $k \in K$. Denoted the traffic flow on path $P \in \mathcal{P}$ by f_P. Let $\mathcal{F}$ be the set of all $\boldsymbol{f} = (f_P)_{P\in\mathcal{P}}$ with $\sum_{P\in\mathcal{P}_k} f_P = d_k$, $k \in K$. Let $f_a \equiv \sum_{P\ni a} f_P$. Let $\ell_a(f_a)$, $a \in A$, be the nondecreasing latency function and $\ell_P(\boldsymbol{f}) \equiv \sum_{a\in P} \ell_a(f_a)$. Define $C^{\boldsymbol{f}}(\boldsymbol{g}) \equiv \sum_{a\in A} \ell_a(f_a)g_a$. A flow $\boldsymbol{f} \in \mathcal{F}$ is said to be a Wardrop *user equilibrium* if $f_P > 0$ and $P \in \mathcal{P}_k$ imply $\ell_P(\boldsymbol{f}) \leq \ell_Q(\boldsymbol{f})$ for all $Q \in \mathcal{P}_k$.

We next move to the mixed strategy formulation. A user is assigned to the OD pair $k \in K$ with the probability $\pi_k = d_k/d$. Denote by $k(P)$ the OD pair served by the path P. Let x_P be the probability that a user with the OD pair $k(P)$ chooses path P. Let $\boldsymbol{x}_k = (x_P)_{P\in\mathcal{P}_k}$ and $\boldsymbol{x} = (\boldsymbol{x}_k)_{k\in K}$. Note that $\boldsymbol{f}_k = d_k\boldsymbol{x}_k$. Denote the set of all strategy profiles by $\mathcal{X}$. The set of mixed strategies of a user with the OD pair k is denoted by $\mathcal{X}_k$. Let $f(\boldsymbol{x}) = (f_P(\boldsymbol{x}))_{P\in\mathcal{P}}$ where $f_P(\boldsymbol{x}) = d_{k(P)}x_P$ and $f_a(\boldsymbol{x}) = \sum_{P\ni a} f_P(\boldsymbol{x})$. The path latency function is $\ell_P(\boldsymbol{x}) = \sum_{a\in P} \ell_a(f_a(\boldsymbol{x}))$. Let $C_k^{\boldsymbol{x}}(\boldsymbol{y}_k) \equiv \sum_{P\in\mathcal{P}_k} \ell_P(\boldsymbol{x})y_P$ and $C^{\boldsymbol{x}}(\boldsymbol{y}) \equiv \sum_{k\in K} \pi_k C_k^{\boldsymbol{x}}(\boldsymbol{y}_k)$.

We now assume that a link $a \in A$ has capacity c_a. Let $\mathcal{F}^c$ be the set of all flow vectors satisfying the demand/link capacity constraints. The set of all strategy profiles satisfying the link capacity constraints is denoted by $\mathcal{X}^c$. Define the following.

1. A path Q is said to be *ex post feasible* w.r.t. $\boldsymbol{f} \in \mathcal{F}^c$ if $f_a < c_a$ for all $a \in Q$. $\boldsymbol{f} \in \mathcal{F}^c$ is an *ex post user equilibrium* if $f_P > 0$ implies $\ell_P(\boldsymbol{f}) \leq \ell_Q(\boldsymbol{f})$ for all ex post feasible paths $Q \in \mathcal{P}_{k(P)}$. We say that $\boldsymbol{x} \in \mathcal{X}^c$ is an *ex post user equilibrium* if $f(\boldsymbol{x})$ is an ex post user equilibrium.
2. $\boldsymbol{y}_k = (y_R) \in \mathcal{X}_k$ is said to be *interim feasible* w.r.t. $\boldsymbol{x} \in \mathcal{X}^c$ if $f_a(\boldsymbol{x}) = c_a$ implies $\sum_{R\ni a, R\in\mathcal{P}_k}(y_R - x_R) \leq 0$. $\boldsymbol{x} = (\boldsymbol{x}_k)_{k\in K} \in \mathcal{X}^c$ is an *interim user equilibrium* if for every $k \in K$, $\boldsymbol{x}_k \in \arg\min_{\boldsymbol{y}_k} C_k^{\boldsymbol{x}}(\boldsymbol{y}_k)$ s.t. $\boldsymbol{y}_k \in \mathcal{X}_k$ is interim feasible w.r.t. $\boldsymbol{x}$.
3. $\boldsymbol{y} \in \mathcal{X}$ is said to be *ex ante feasible* w.r.t. $\boldsymbol{x} \in \mathcal{X}^c$ if $f_a(\boldsymbol{x}) = c_a$ implies $\sum_{R\ni a} \pi_{k(R)}(y_R - x_R) \leq 0$. $\boldsymbol{x} \in \mathcal{X}^c$ is an *ex ante user equilibrium* if $\boldsymbol{x} \in \arg\min_{\boldsymbol{y}} C^{\boldsymbol{x}}(\boldsymbol{y})$ s.t. $\boldsymbol{y} \in \mathcal{X}$ is *ex ante* feasible w.r.t. $\boldsymbol{x}$.

Let $\mathcal{E}_{\text{ex post}}$, $\mathcal{E}_{\text{interim}}$ and $\mathcal{E}_{\text{ex ante}}$ represent the sets of *ex post*, interim, and *ex ante* user equilibriums, resp. We consider the following constrained BMW optimization problem.

$$\min_{\boldsymbol{x}} \theta(\boldsymbol{x}) \text{ s.t. } \boldsymbol{x} \in \mathcal{X}^c \tag{1}$$

where $\theta(\boldsymbol{x}) = \sum_{a\in A} \int_0^{f_a(\boldsymbol{x})} \ell_a(u)du.$. Here are the main results of this paper.

1. $\mathcal{E}_{\text{ex post}} \supseteq \mathcal{E}_{\text{interim}} \supseteq \mathcal{E}_{\text{ex ante}}$
2. $\boldsymbol{x} \in \mathcal{E}_{\text{ex ante}}$ iff it solves (1).

Thus, the ex ante equilibrium is a characterization of the constrained BMW optimization.

[1] Beckmann M, McGuire C, Winsten CB, 1956. Studies in the Economics of Transportation. Yale Univ. Press.

[2] Correa JR, Schulz AS, Stier-Moses NE, 2004. Math OR 29 961–976.

[3] Marcotte P, Nguyen S, Schoeb A, 2004 Oper. Res. 52 191–212.