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A well-defined extended production possibility set and strongly monotonic efficiency measures

01206313 成蹊大学 *関谷 和之 SEKITANI Kazuyuki
05000602 東京理科大学 趙 宇 ZHAO Yu

1. Introduction

Recently, the concept of similarity of the projection point has been incorporated into the studies of least-distance DEA. The least-distance projection point is known as the closest target of the evaluated DMU. The extended facet production possibility set (EFPPS) is often used in the least-distance DEA to ensure monotonicity. However, previous studies have shown that the EFPPS is not well-defined if no facet exists on the efficient frontier. This study further show that the EFPPS (i) is unsuitable for the DEA models that have nonlinear objective functions; (ii) may also result in weakly monotonic efficiencies in the input-oriented measurement. We introduce a well-defined extended production possibility set based on weight restrictions and further propose a generalized output-oriented DEA model that (i) satisfies strong monotonicity; (ii) provides the closest target; (iii) can handle nonlinear objective functions and can be solved using linear programming (LP).

2. Issues of input-oriented models

The following counterexample shows that the EFPPS may result in weakly monotonic efficiencies in the input-oriented measurement. We proceed with the discussion under the assumption of various returns-to-scale.

Table 1: A numeric example

	A	B	C	D	E
x	1	2	5	3	4
y	4	5	6	2	2

The efficient DMUs consist of DMU_A, DMU_B, and DMU_C. The strongly frontier of the PPS consists of two line segments, that is

$$\begin{aligned}\partial^s(P) = &\{(x,y) \mid x-y+3=0, 1\le x\le 2\} \\ &\cup \{(x,y) \mid x-3y+13=0, 2\le x\le 5\}.\end{aligned}$$

The EFPPS can be represented as

$$P_{\scriptscriptstyle EXFA} = P_{\scriptscriptstyle CON} \cap \mathbb{R}^2_+,$$

where

$$P_{\scriptscriptstyle CON} = \left\{ (x,y) \;\middle|\; \begin{array}{l} x-y+3\ge 0, \\ x-3y+13\ge 0 \end{array} \right\}$$

is a convex PPS. The strongly and weakly efficient frontiers of $P_{\scriptscriptstyle EXFA}$ are given by

$$\begin{aligned}&\partial^s(P_{\scriptscriptstyle EXFA}) = \partial^s(P_{\scriptscriptstyle CON}) \cap \mathbb{R}^2_+, \\ &\partial^w(P_{\scriptscriptstyle EXFA}) = \partial^s(P_{\scriptscriptstyle EXFA}) \cup \{(0,y) \mid 0\le y\le 3\},\end{aligned}$$

respectively. Projecting the input x in $P_{\scriptscriptstyle EXFA}$ towards $\partial^s(P_{\scriptscriptstyle EXFA})$ may lead to infeasible solutions (e.g., $\min\{\,|\delta^-| \mid (x_D-\delta^- x_D, y_D)\in\partial^s(P_{\scriptscriptstyle EXFA})\,\}$). Since there exists $(x,y)\in\partial^w(P_{\scriptscriptstyle EXFA})$ for any $y\ge 0$, we consider the following commensurable Hölder input distance function for the measurement of inefficiency:

$$D^-_{\scriptscriptstyle EXFA}(x,y) = \min\left\{\,|\delta^-| \;\middle|\; (x-\delta^- x, y)\in\partial^w(P_{\scriptscriptstyle EXFA})\,\right\}.$$

If $(x,y)\in P_{\scriptscriptstyle EXFA}$ and $y\le 3$, $D^-_{\scriptscriptstyle EXFA}(x,y)$ projects it onto $\{(0,y)\mid 0\le y\le 3\}\subseteq\partial^w(P_{\scriptscriptstyle EXFA})$. Therefore, it follows from $y_D\le 3$ and $y_E\le 3$ that $D^-_{\scriptscriptstyle EXFA}(x_D,y_D) = D^-_{\scriptscriptstyle EXFA}(x_E,y_E) = 1$, which implies that strong monotonicity is violated.

3. A well-defined extended production possibility set

It is well-known that the EFPPS is not well-defined if there is no facet on the strongly efficient frontier. We show that this issue can be effectively avoided by using a proper matrix A. Let A be a $q\times(m+s)$ positive matrix and consider an input-output weight restriction

$$(\boldsymbol{v},\boldsymbol{u}) = \boldsymbol{p}A \text{ and } \boldsymbol{p}\ge\boldsymbol{0}. \qquad (1)$$

Theorem 3.1. *For any positive* $\epsilon < \frac{1}{(m+s)^2}$*, the restrictions* $v_1\ge\epsilon, \cdots v_m\ge\epsilon$*,* $u_1\ge\epsilon,\cdots,u_s\ge\epsilon$ *and* $\sum_{i=1}^m v_i + \sum_{r=1}^s u_r = 1$ *satisfy* (1) *for a postive matrix* $A = (I-\epsilon(m+s)E)^{-1}$*, where* I *is an identity matrix with the size* $(m+s)$ *and* E *is all-one matrix with the size* $(m+s)$*.*

For any efficient input-output vector $(\boldsymbol{x}, \boldsymbol{y})$, there exists a hyperplane

$$\boldsymbol{v}\boldsymbol{x} + v_0 - \boldsymbol{u}\boldsymbol{y} = 0 \tag{2}$$

with a positive normal vector $(\boldsymbol{v}, \boldsymbol{u})$. Without loss of generality we can assume $\sum_{i=1}^m v_i + \sum_{r=1}^s u_r = 1$ on (2). By using a small number $\epsilon > 0$, DEA usually replaces the positive conditions $v_i > 0$ and $u_r > 0$ with the non-Archimedean conditions $v_i \geq \epsilon$ and $u_r \geq \epsilon$. It follows from Theorem 3.1 that the non-Archimedean condition is the input-output weight restrictions (1).

The matrix A can be also interpreted by the production trade-offs between the inputs and outputs. Based on the weight restrictions (1), we define the extended production possibility set P_A as

$$\left\{ (\boldsymbol{x}, \boldsymbol{y}) \;\middle|\; \begin{array}{ll} \sum_{j=1}^n \lambda_j \boldsymbol{x}_j + \boldsymbol{d}^- = \boldsymbol{x}, & A\binom{\boldsymbol{d}^-}{\boldsymbol{d}^+} \geq \boldsymbol{0} \\ \sum_{j=1}^n \lambda_j \boldsymbol{y}_j - \boldsymbol{d}^+ = \boldsymbol{y}, & \boldsymbol{y} \geq \boldsymbol{0} \\ \sum_{j=1}^n \lambda_j = 1, & \boldsymbol{\lambda} \geq \boldsymbol{0} \end{array} \right\},$$

and it has properties as follows:

Theorem 3.2. *The PPS P_A satisfies $\partial^w(P_A) = \partial^s(P_A)$ and $\{ \boldsymbol{\eta} \mid (\boldsymbol{x}, \boldsymbol{\eta}) \in P_A \}$ is bounded for any $(\boldsymbol{x}, \boldsymbol{y}) \in P_A$.*

4. A generalized output-oriented DEA model

The output-oriented SBM DEA model [4] is

$$\begin{array}{lll} \min & \dfrac{1}{1 + \frac{1}{s}\sum_{r=1}^s \delta_r} & (3) \\ \text{s.t.} & (\boldsymbol{x}, \boldsymbol{y} + N(\boldsymbol{y})\boldsymbol{\delta}) \in P_A, \ \boldsymbol{\delta} \geq \boldsymbol{0}, & (4) \end{array}$$

where $N(\boldsymbol{y})$ is a diagonal matrix whose (r, r) entry is y_r. The output-oriented RM DEA model [2] is

$$\min \left\{ \frac{1}{s} \sum_{r=1}^s \frac{1}{1 + \delta_r} \;\middle|\; (4) \right\}, \tag{5}$$

which is equivalent to the output-oriented BRWZ DEA model [1]. The output-oriented GDF DEA model [3] is

$$\min \left\{ \left(\prod_{r=1}^s \frac{1}{1 + \delta_r} \right)^{1/s} \;\middle|\; (4) \right\}. \tag{6}$$

All objective functions of the DEA models are decreasing, continuous, and quasi-convex on $\mathbb{R}^s_+$.

Let g be decreasing, continuous, and quasi-convex on $\mathbb{R}^s_+$ and consider the following output-oriented DEA model, where the best practice is on $\partial^s(P_A)$:

$$\begin{array}{lll} \max & g(\boldsymbol{\delta}) & (7) \\ \text{s.t.} & (\boldsymbol{x}, \boldsymbol{y} + N(\boldsymbol{y})\boldsymbol{\delta}) \in \partial^s(P_A), \ \boldsymbol{\delta} \geq \boldsymbol{0}. & (8) \end{array}$$

Let $f_A(\boldsymbol{x}, \boldsymbol{y})$ be the optimal value of the DEA model (7)–(8).

Theorem 4.1. *The efficiency measure f_A is strongly monotonic on $P_A \cap \left(\left(\mathbb{R}^m_+ \setminus \{\boldsymbol{0}\} \right) \times \mathbb{R}^s_{++} \right)$.*

The commensurable Hölder output distance function for the measurement of inefficiency is defined as $\min \{ \sum_{r=1}^s \delta_r \mid (8) \}$. Let $D_A(\boldsymbol{x}, \boldsymbol{y})$ be the commensurable Hölder output distance function for $(\boldsymbol{x}, \boldsymbol{y})$. Then, we have $D_A(\boldsymbol{x}, \boldsymbol{y}) = \min \{ \delta_1^*, \ldots, \delta_s^* \}$, and δ_r^* can be obtained by solving the following LP problem: $\max \{ \delta \mid (\boldsymbol{x}, \boldsymbol{y} + y_r \delta \boldsymbol{e}_r) \in P_A \}$ for all $r = 1, \ldots, s$. By choosing $g(\boldsymbol{\delta})$ from the SBM DEA model (3)–(4), RM (BRWZ) DEA model (5), and GDF DEA model (6), the output-oriented maximum efficiency measure can be a decreasing function of $D_A(\boldsymbol{x}, \boldsymbol{y})$ as follows: $f_A(\boldsymbol{x}, \boldsymbol{y}) =$

$$\begin{cases} \frac{1}{1 + \frac{D_A(\boldsymbol{x}, \boldsymbol{y})}{s}} & \text{if } g(\boldsymbol{\delta}) = \frac{1}{1 + \frac{1}{s}\sum_{r=1}^s \delta_r} \\ \frac{1}{s}\left(s - \frac{D_A(\boldsymbol{x}, \boldsymbol{y})}{1 + D_A(\boldsymbol{x}, \boldsymbol{y})} \right) & \text{if } g(\boldsymbol{\delta}) = \frac{1}{s}\sum_{r=1}^s \frac{1}{1 + \delta_r} \\ \left(\frac{1}{1 + D_A(\boldsymbol{x}, \boldsymbol{y})} \right)^{\frac{1}{s}} & \text{if } g(\boldsymbol{\delta}) = \left(\prod_{r=1}^s \frac{1}{1 + \delta_r} \right)^{1/s} \end{cases}.$$

References

[1] Brockett, P. L., Rousseau, J. J., Wang, Y., & Zhow, L. (1997). Implementation of DEA models using GAMS. Research Report 765.

[2] Färe, R., Grosskopf, S., & Lovell, C. A. (1985). Nonradial Efficiency Measures. In The Measurement of Efficiency of Production (pp. 141-162). Springer, Dordrecht.

[3] Portela, M. C. A. S., & Thanassoulis, E. (2005). Profitability of a sample of Portuguese bank branches and its decomposition into technical and allocative components. European Journal of Operational Research, 162(3), 850-866.

[4] Tone, K. (2001). A slacks-based measure of efficiency in data envelopment analysis. European journal of operational research, 130(3), 498-509.